Made by ALEX

P1 Chapter 6.1

«The Rule for Differentiation»

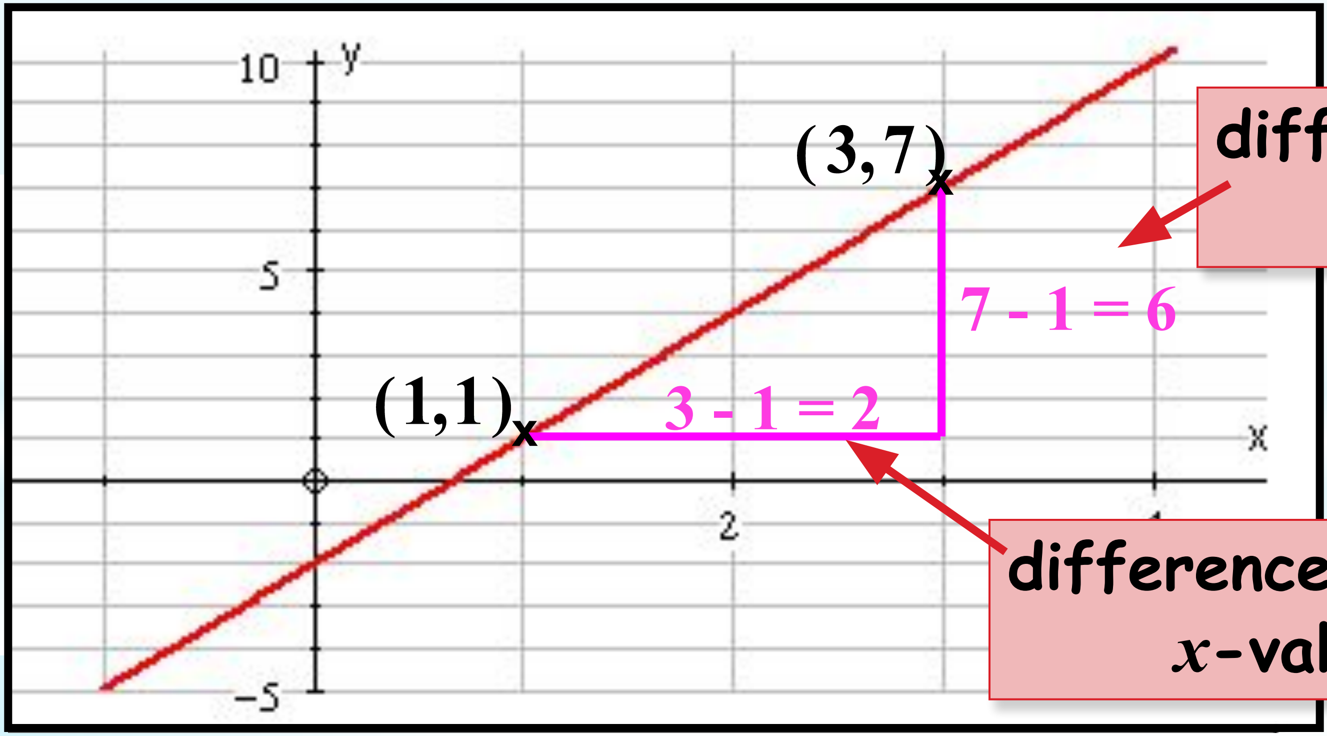
The Gradient of a Straight Line

The gradient of a straight line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are points on the line e.g. Find the gradient of the line joining the points with coordinates $(1,1)$ and $(3,7)$

Solution: $\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$ $\Rightarrow$ $m = \frac{7-1}{3-1} = \frac{6}{2} = 3$



difference in the y-values

difference in the x-values

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$$m = \frac{\text{the difference in the } y\text{-values}}{\text{the difference in the } x-\text{values}}$$

We use this idea to get the gradient at a point on a curve

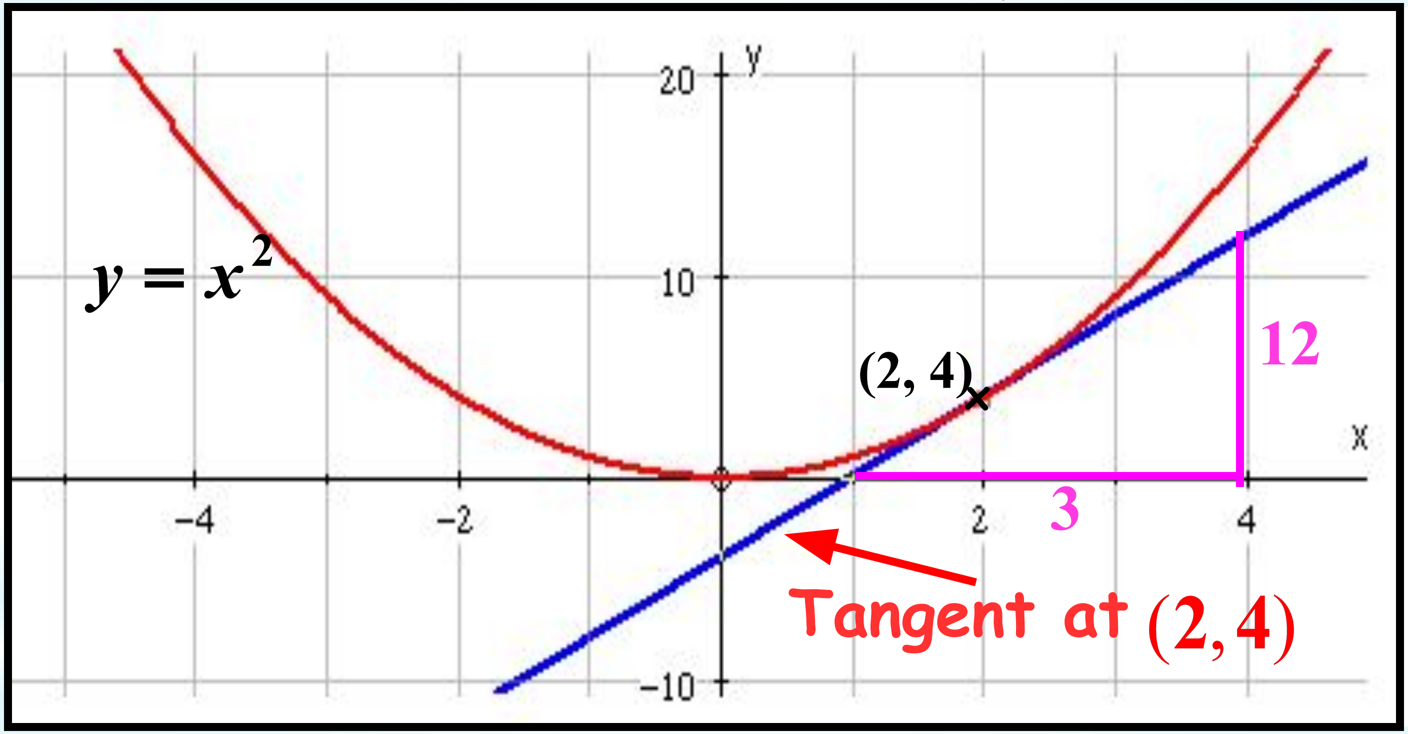
Gradients are important as they measure the rate of change of one variable with another. For the graphs in this section, the gradient measures how y changes with x

This branch of Mathematics is called Calculus

The Gradient at a point on a Curve

Definition: The gradient of a point on a curve equals the gradient of the tangent at that point.

e.g.

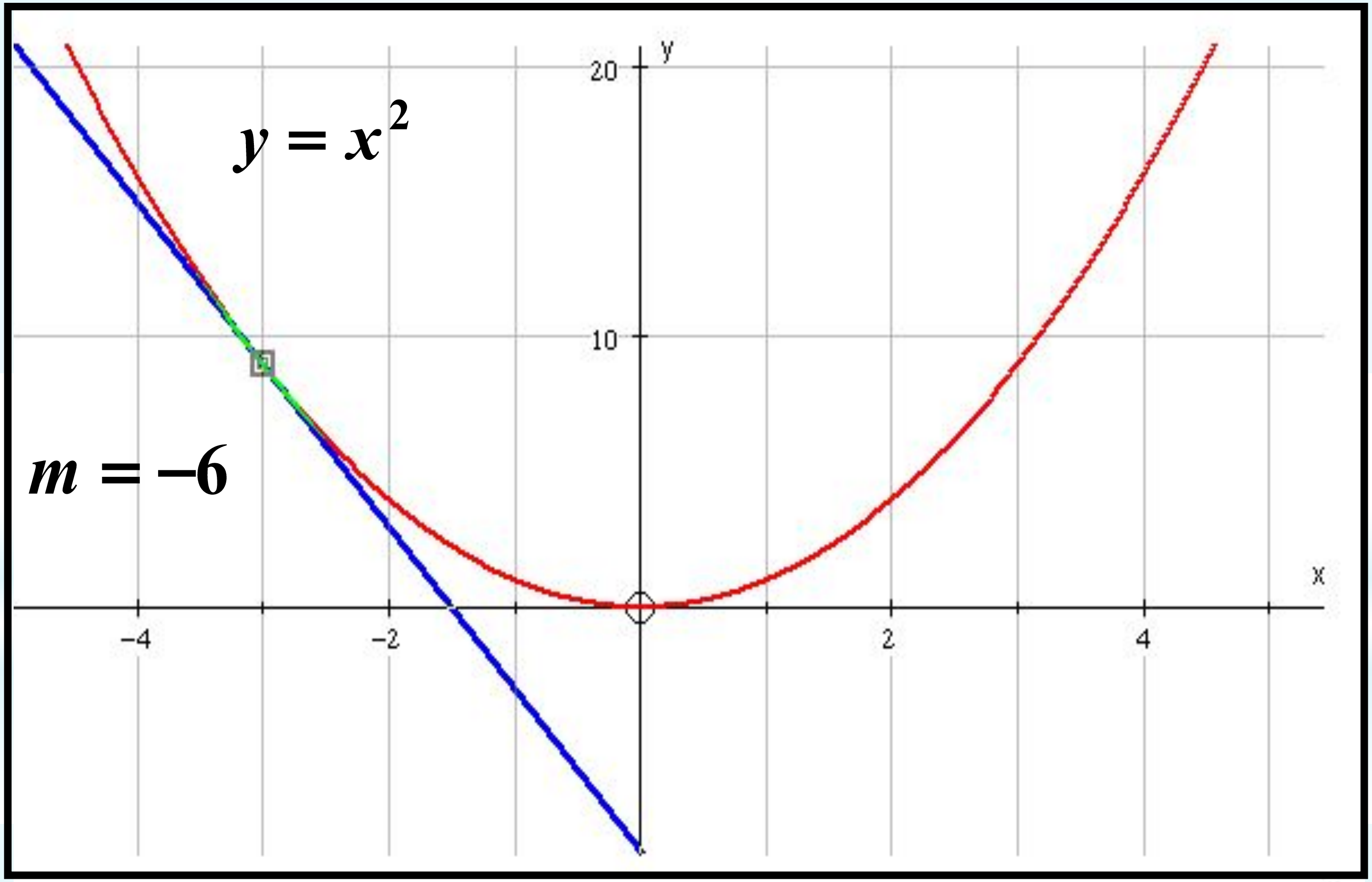


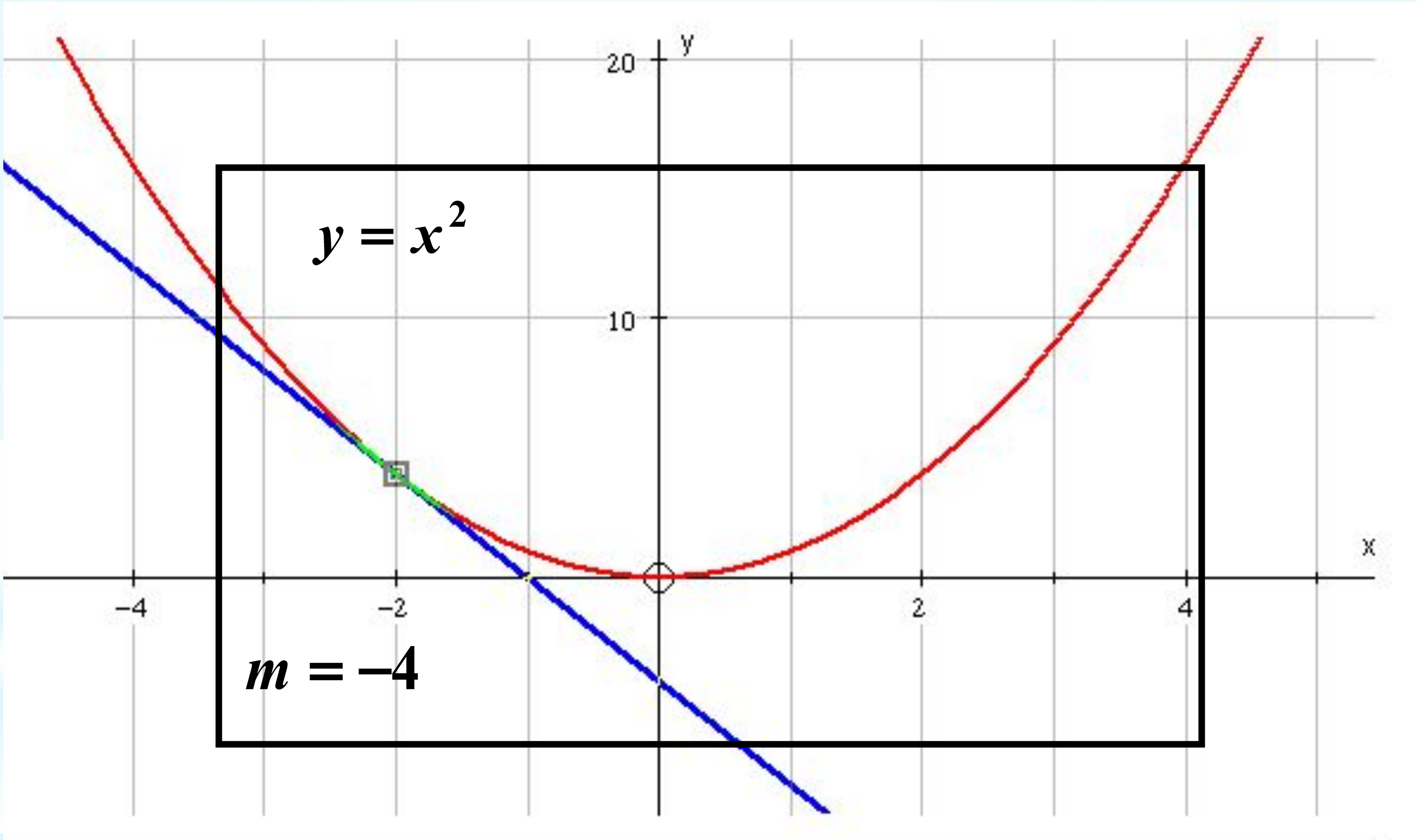
The gradient of the tangent at (2, 4) is $m = \frac{12}{3} = 4$

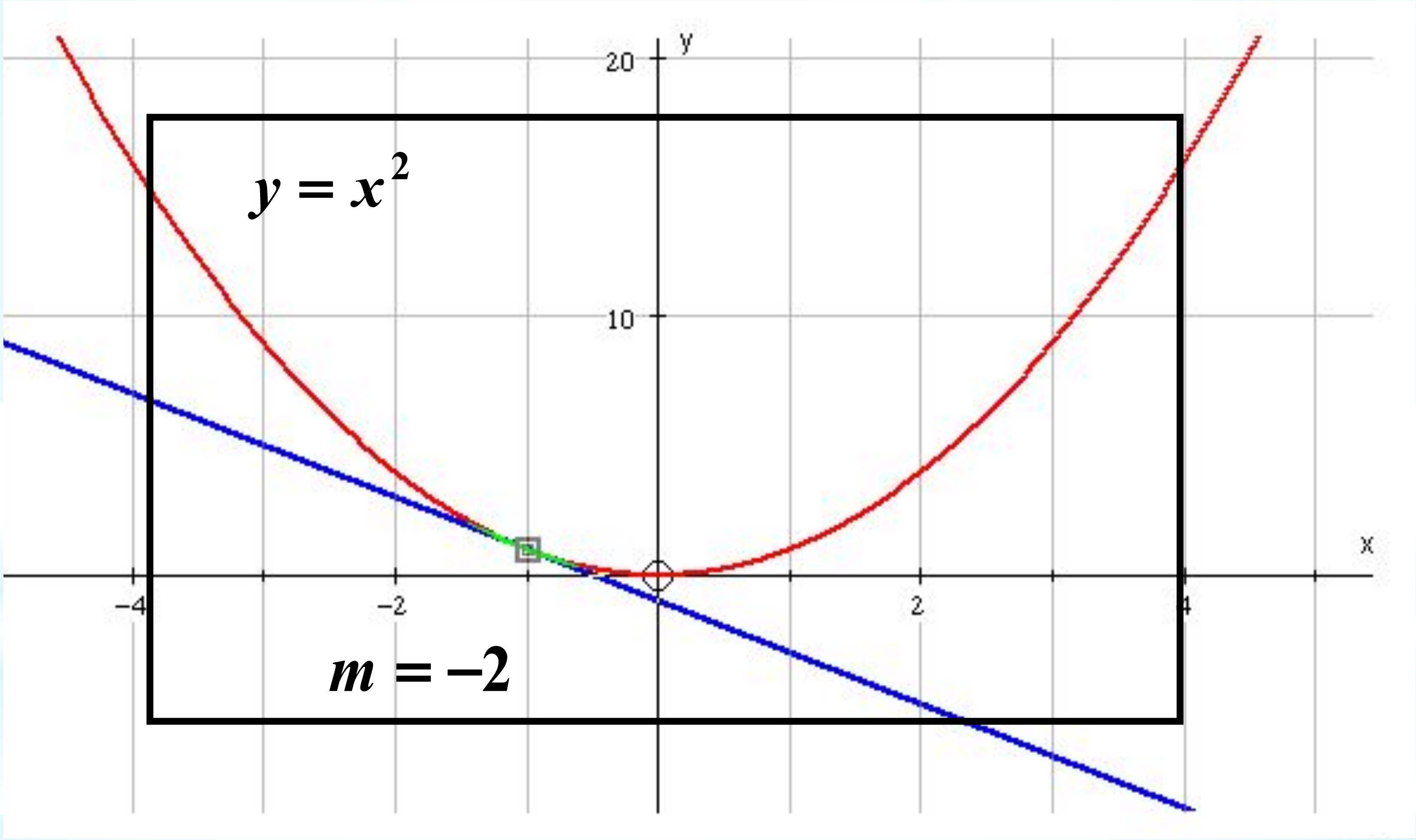
So, the gradient of the curve at (2, 4) is 4

The gradient changes as we move along a curve

e.g.




y
20
10
x
−4
−2
2
4
$y = x^2$
$m = -4$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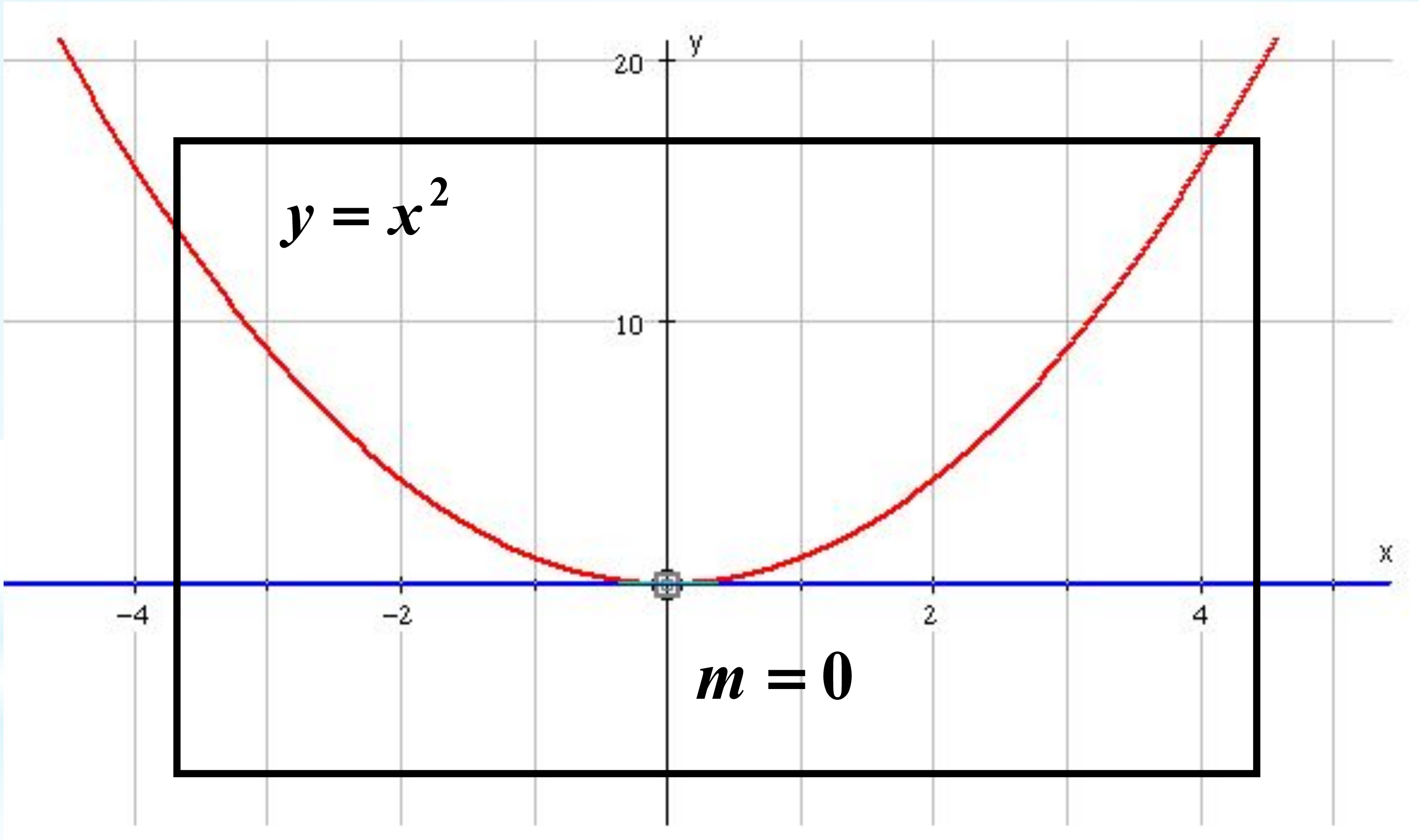

y
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x
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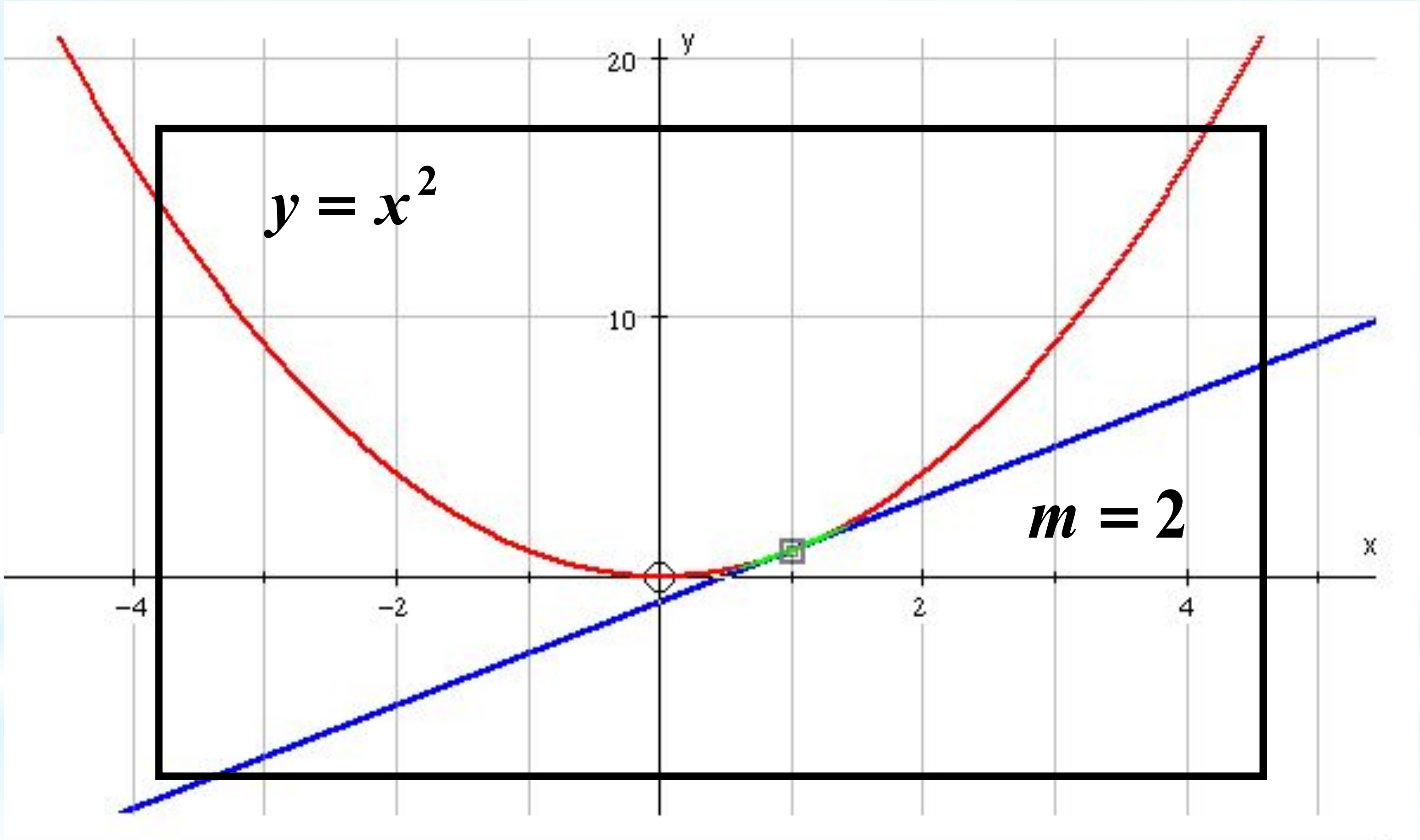
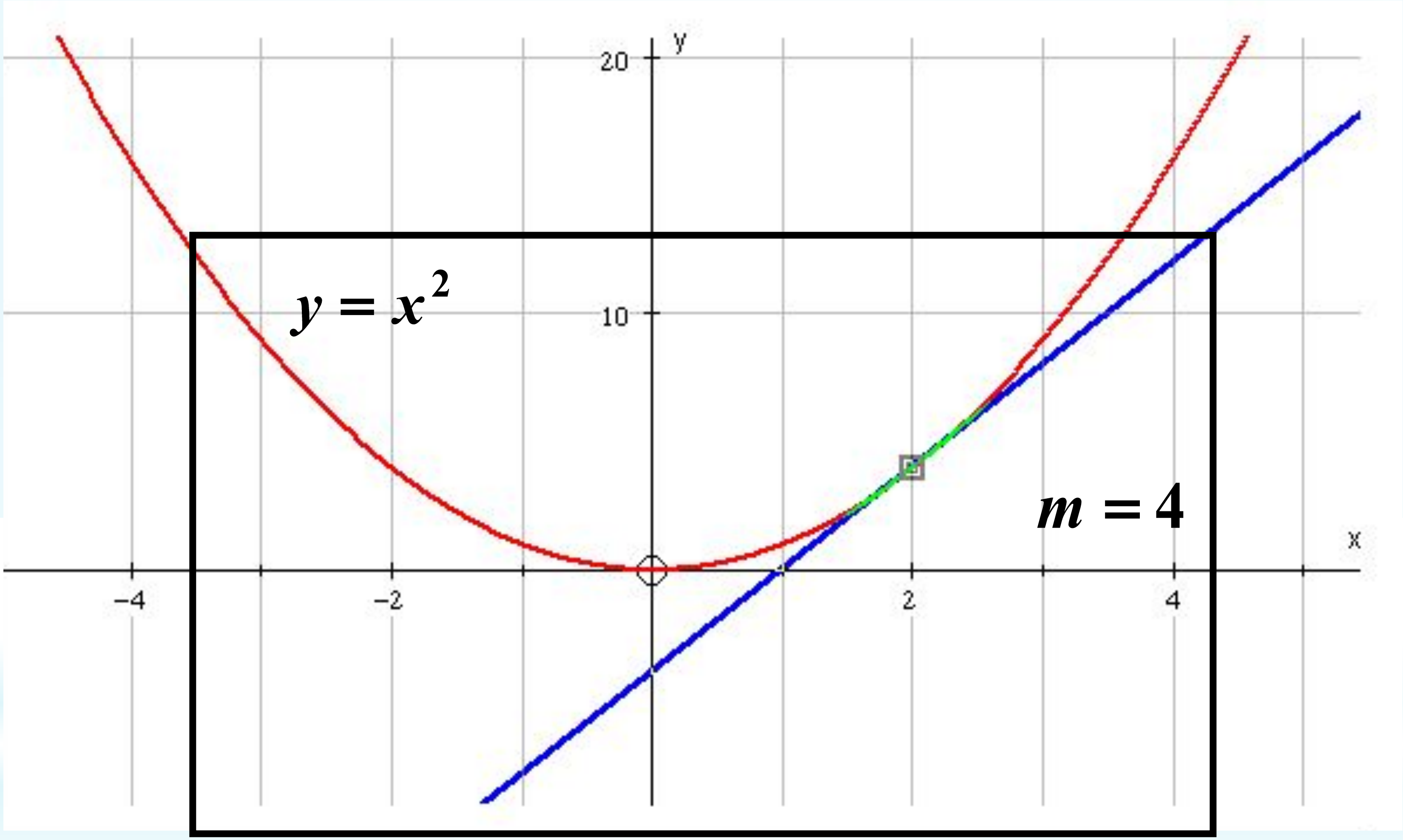
4
$y = x^2$
$m = 0$


y
20
10
$y = x^2$
$m = 2$
x
−4
−2
2
4



y
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10
x
−4
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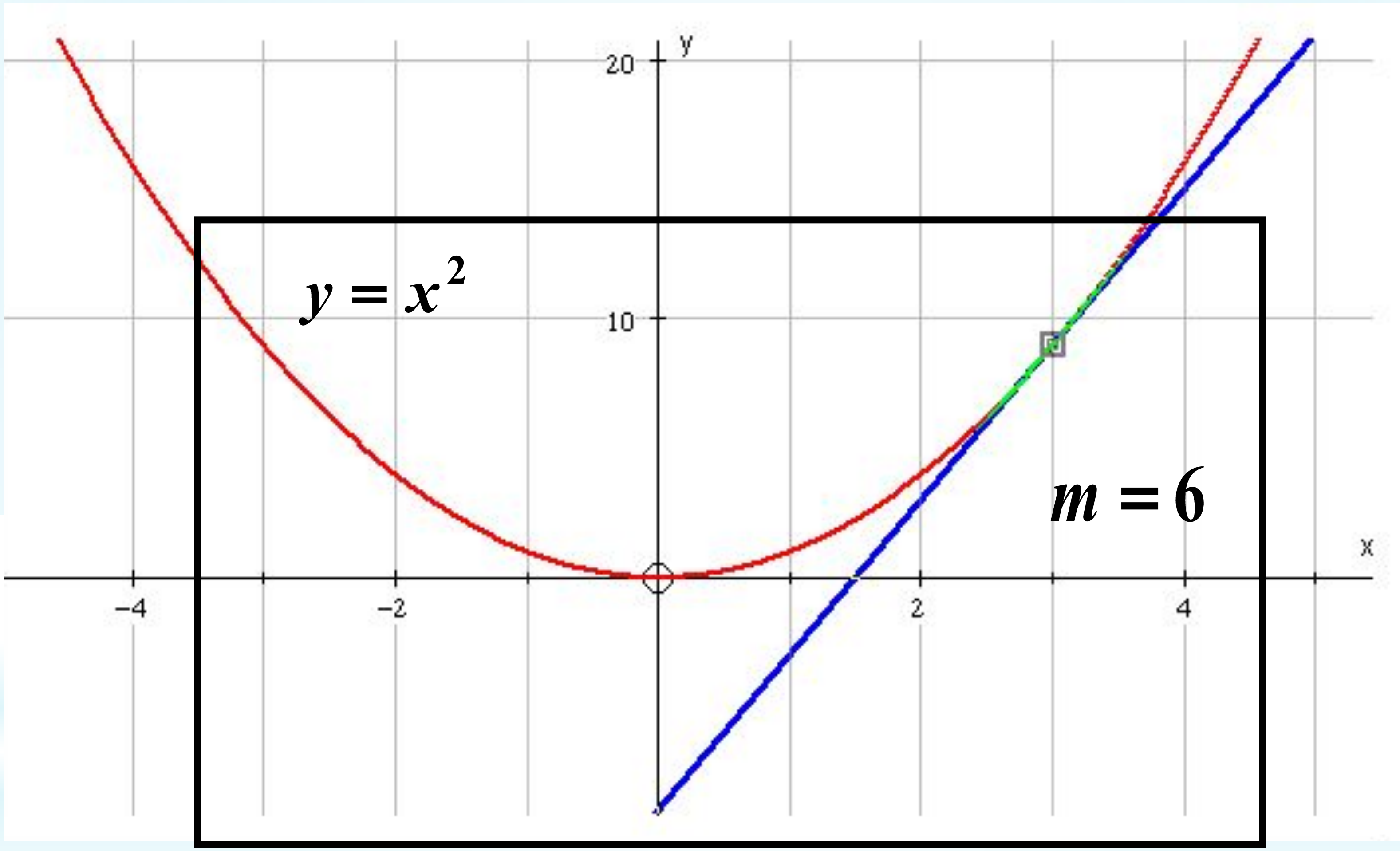
2
4
$y = x^2$
$m = 4$



y
20
10
x
−4
−2
2
4
$y = x^2$
$m = 6$

For the curve $y = x^2$ we have the following gradients:

Point on the curve	Gradient
$(-3, 9)$	-6
$(-2, 4)$	-4
$(-1, 1)$	-2
$(1, 1)$	2
$(2, 4)$	4
$(3, 9)$	6

At every point, the gradient is twice the x-value

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At every point on $y = x^2$ the gradient is twice the x-value

This rule can be written as $\frac{dy}{dx} = 2x$

The notation comes from the idea of the gradient of a line being

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$$\frac{\text{the } \mathbf{d}\text{ifference in the } y \text{ -values}}{\text{the } \mathbf{d}\text{ifference in the } x \text{ - values}}$$

$\dfrac{dy}{dx}$ is read as " dy by dx "

The function giving the gradient of a curve is called the gradient function

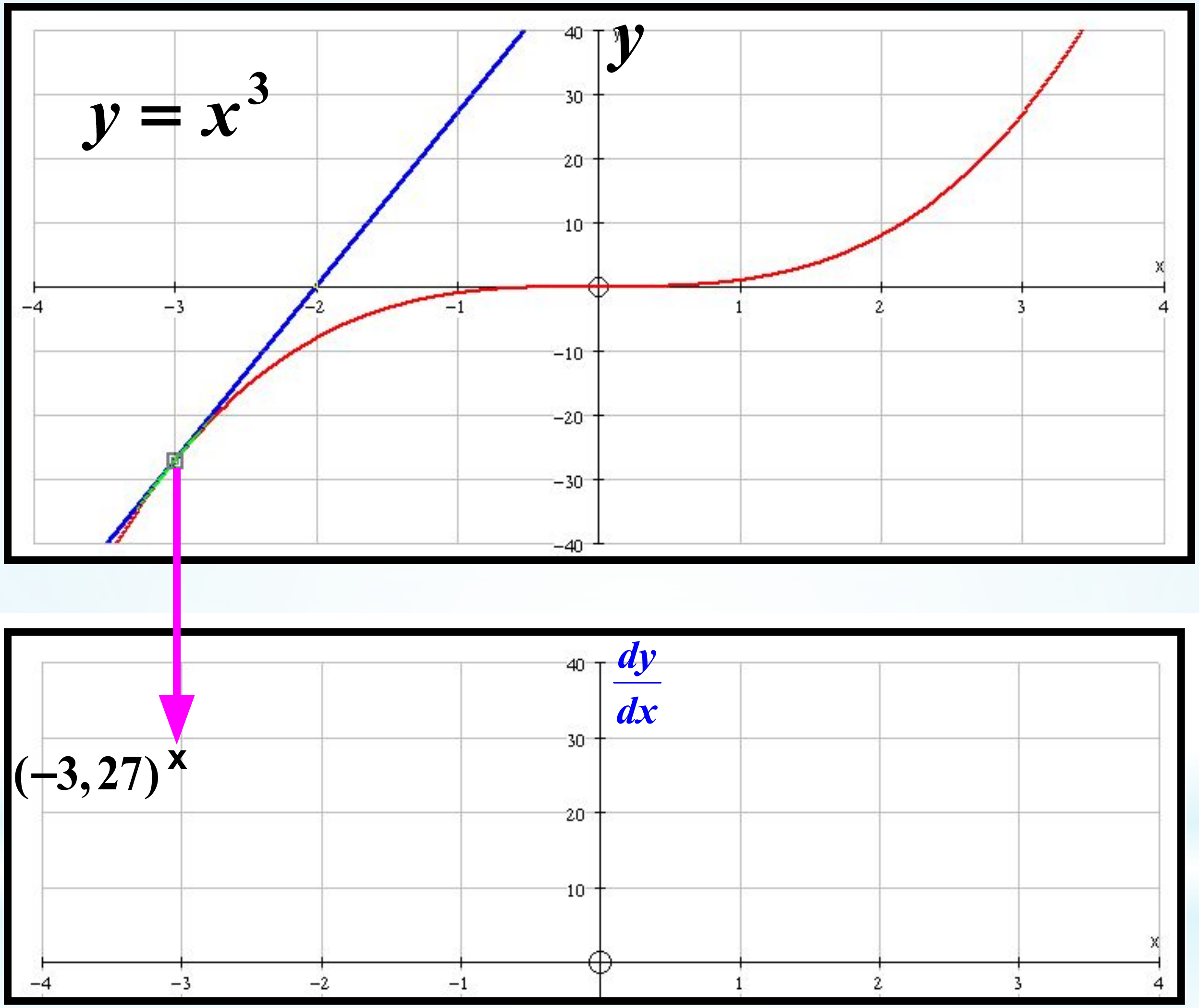
Other curves and their gradient functions

$$y = x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2$$

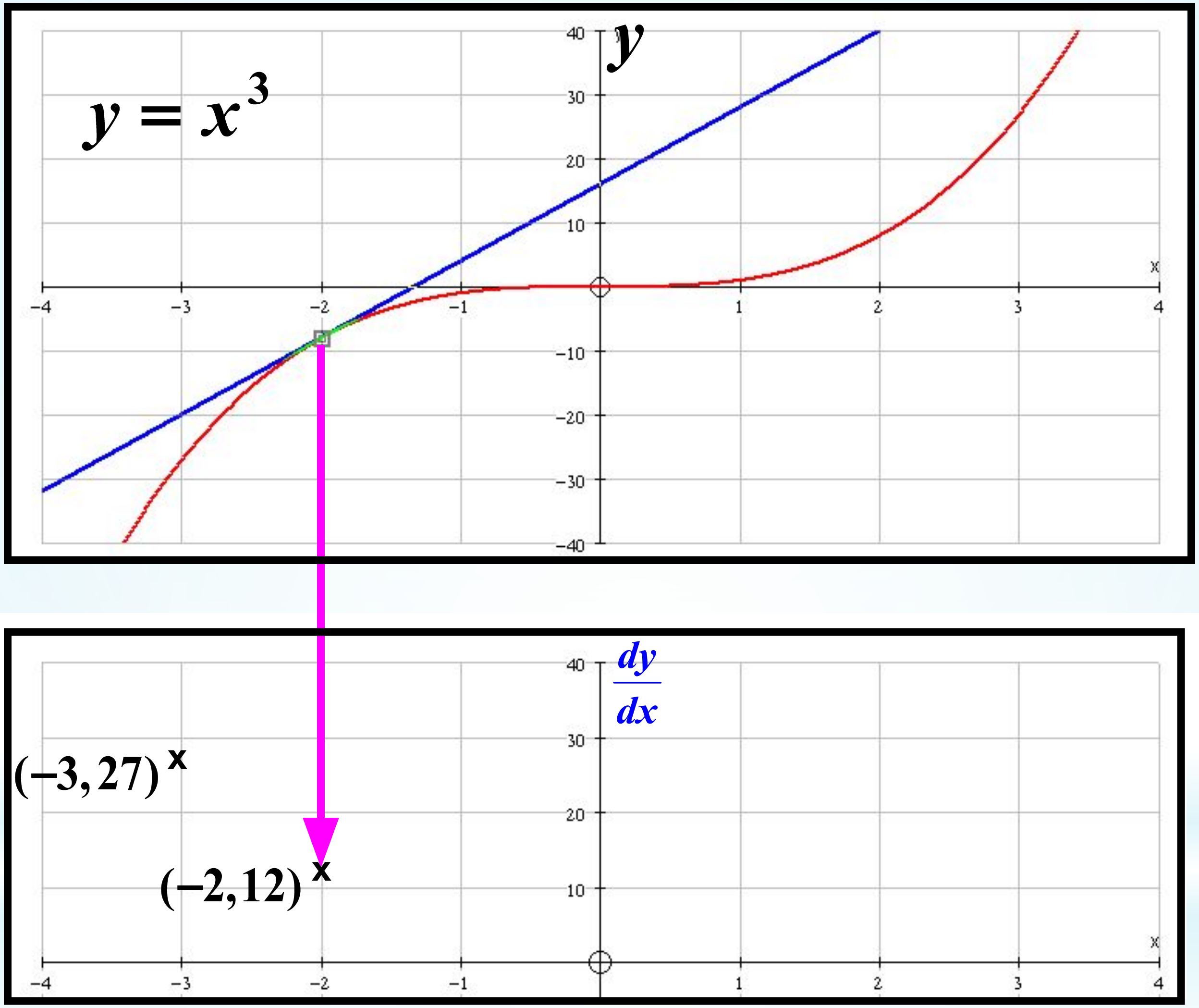
The rule for the gradient function of a curve of the form $y = x^n$ is $\frac{dy}{dx} = nx^{n-1}$

- "power to the front and multiply"
- "subtract 1 from the power"

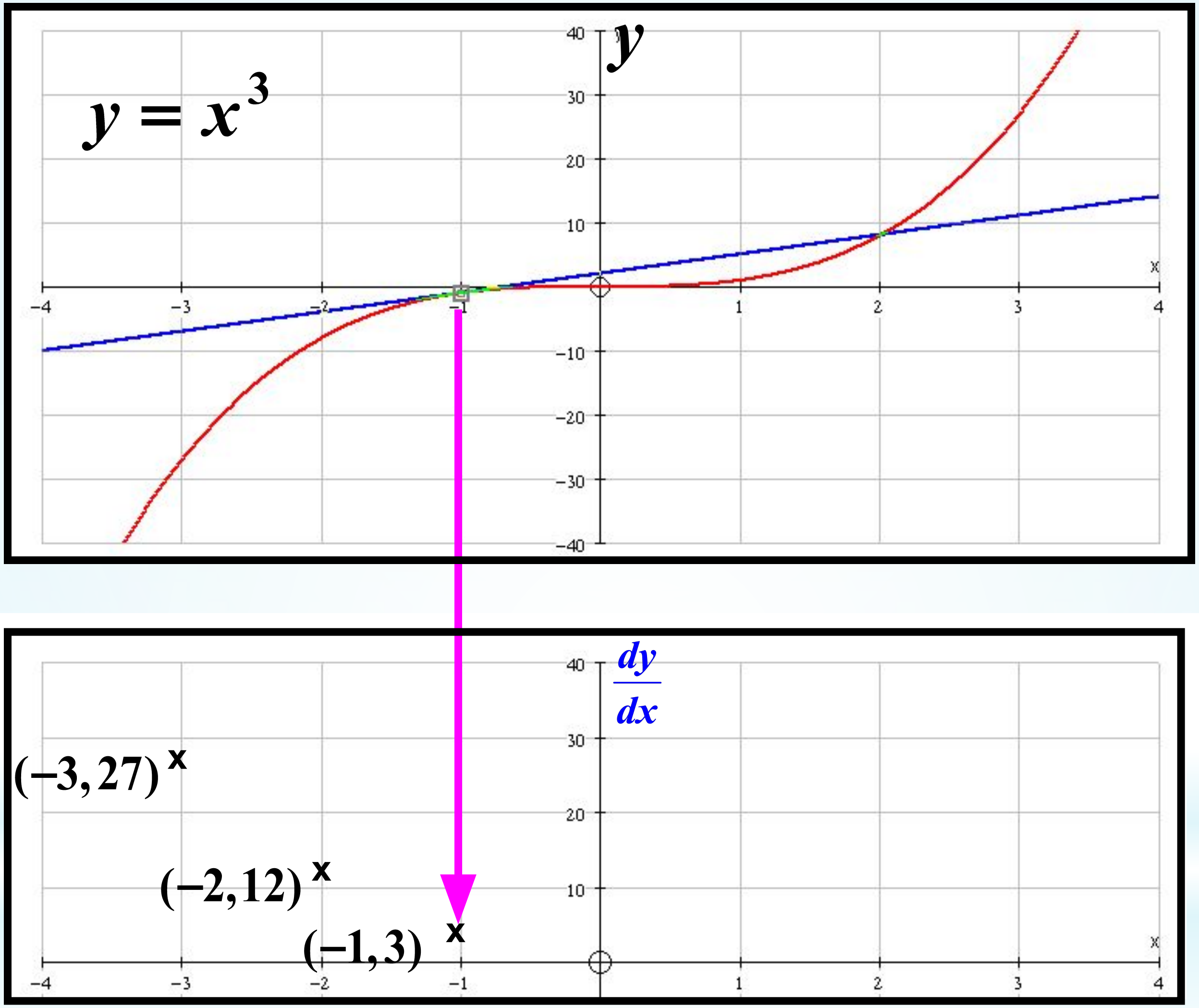
Although this rule won't be proved, we can illustrate it for $y = x^3$ by sketching the gradients at points on the curve



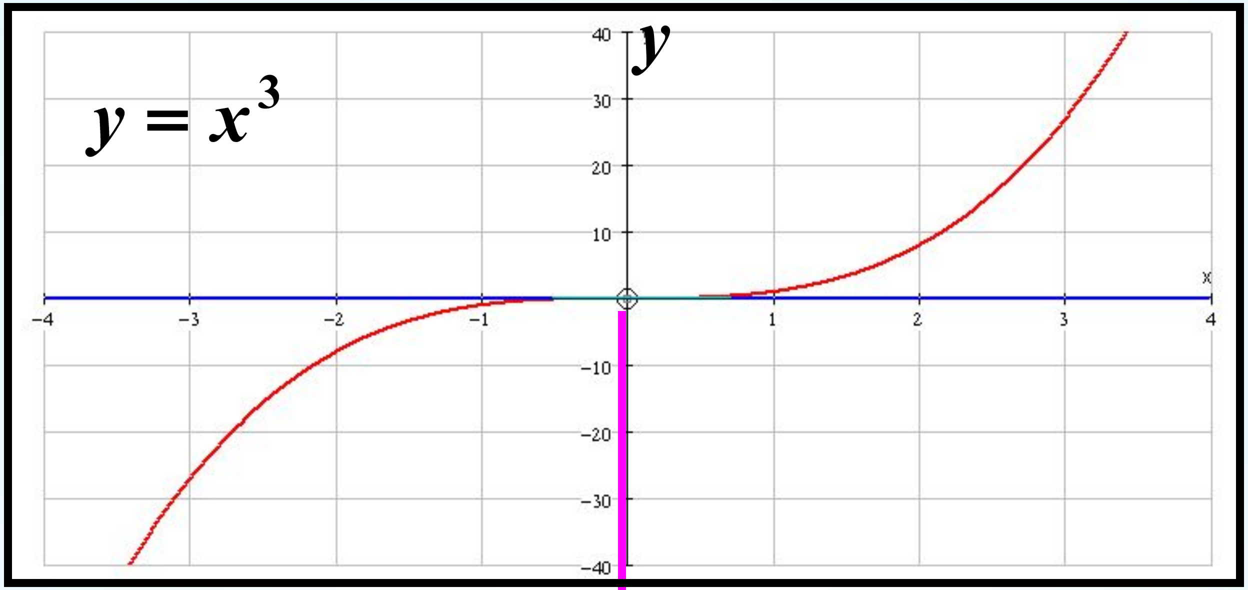
Gradient of $y = x^3$

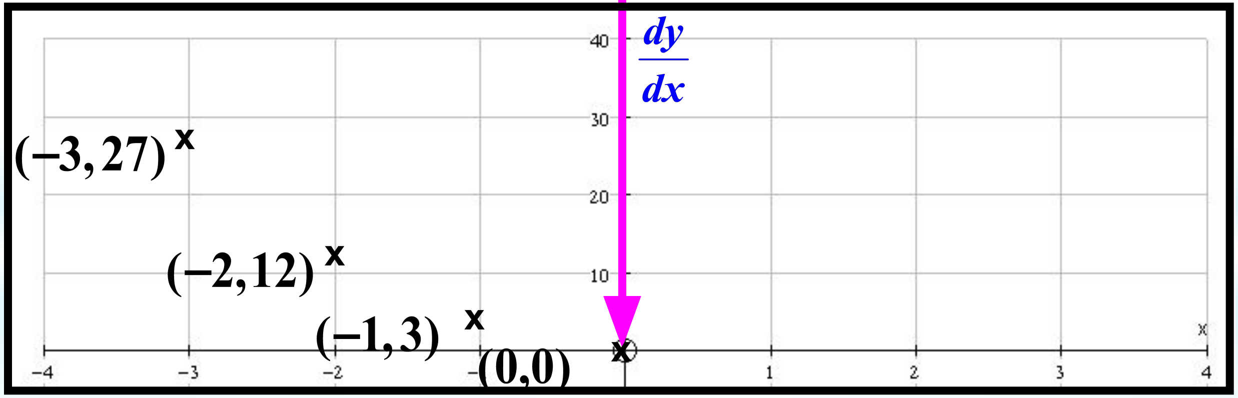


Gradient of $y = x^3$

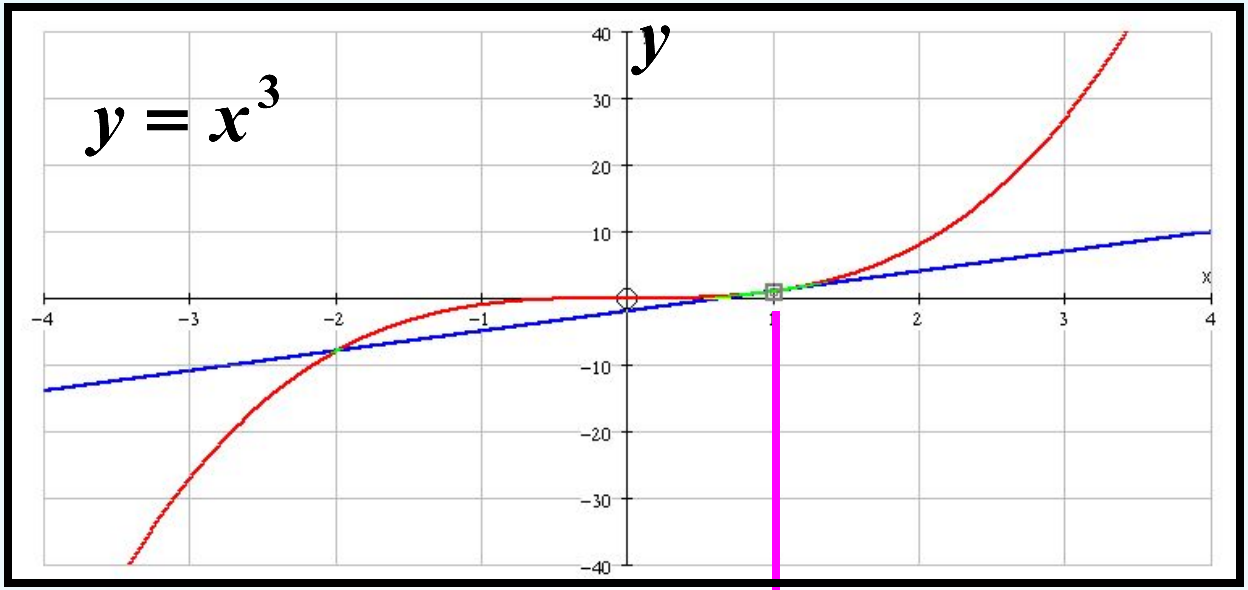


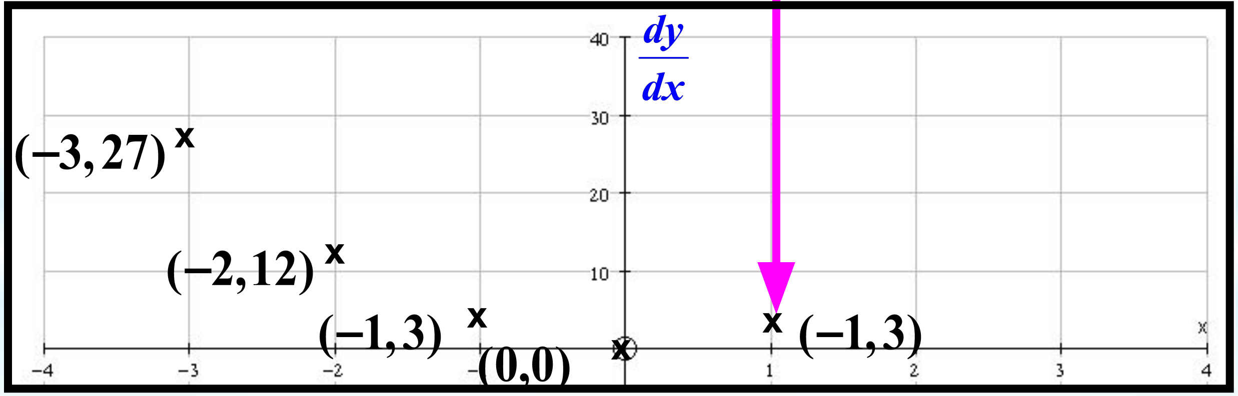
Gradient of $y = x^3$



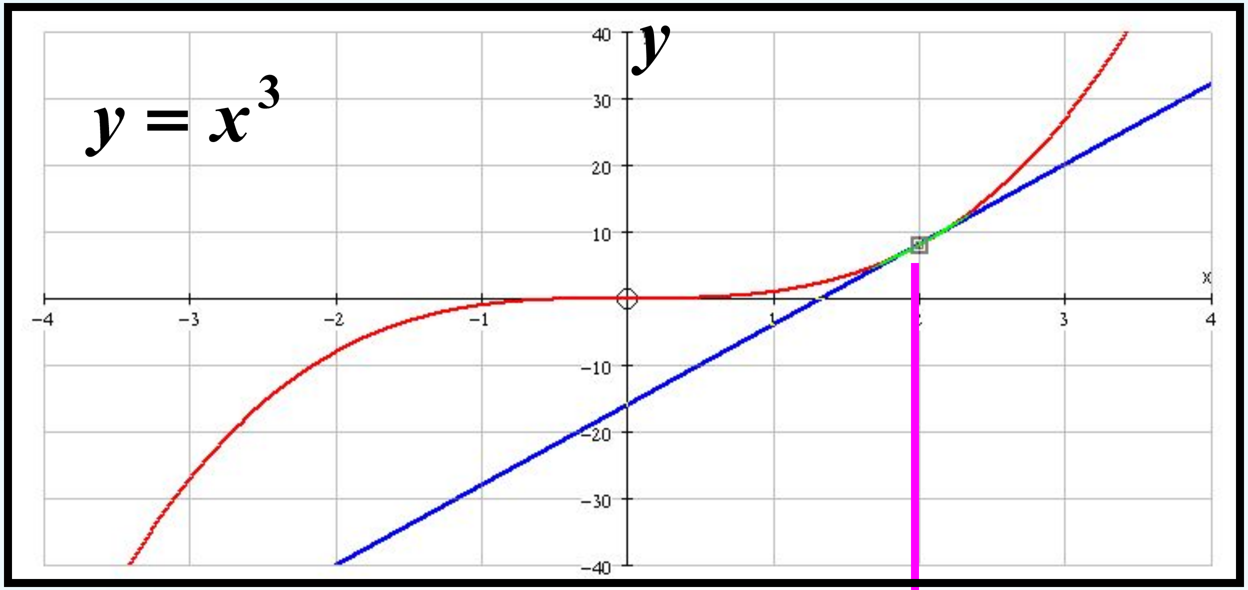


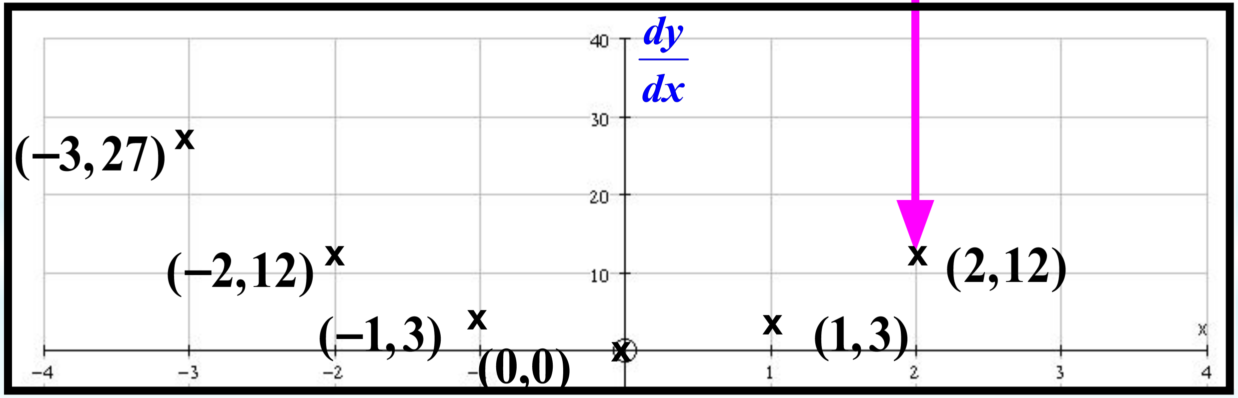
Gradient of $y = x^3$





Gradient of $y = x^3$





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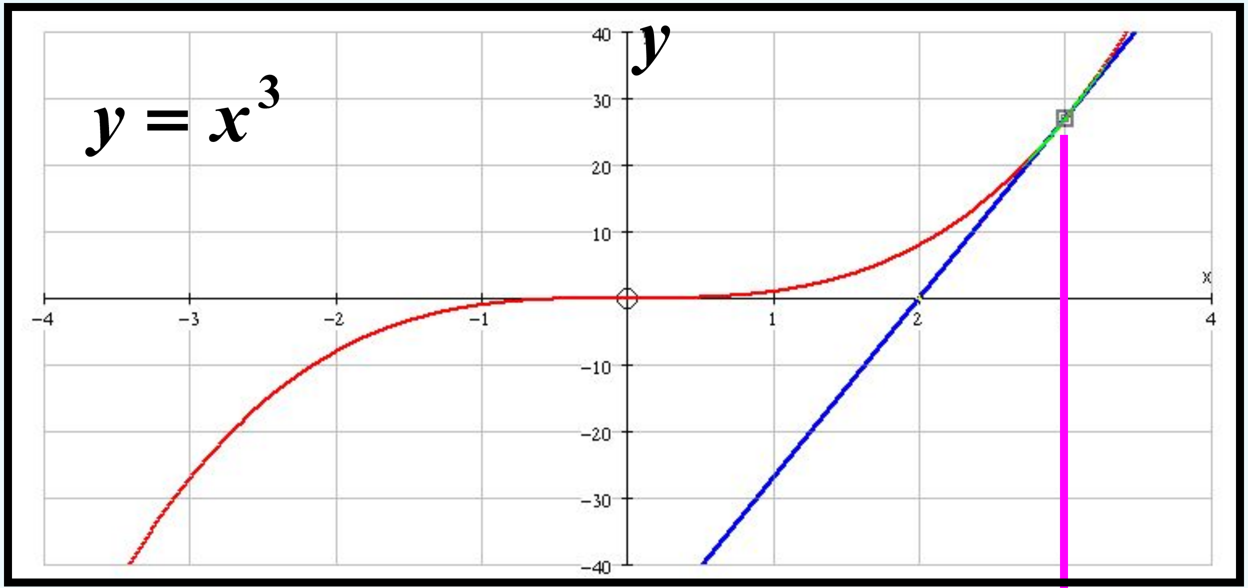

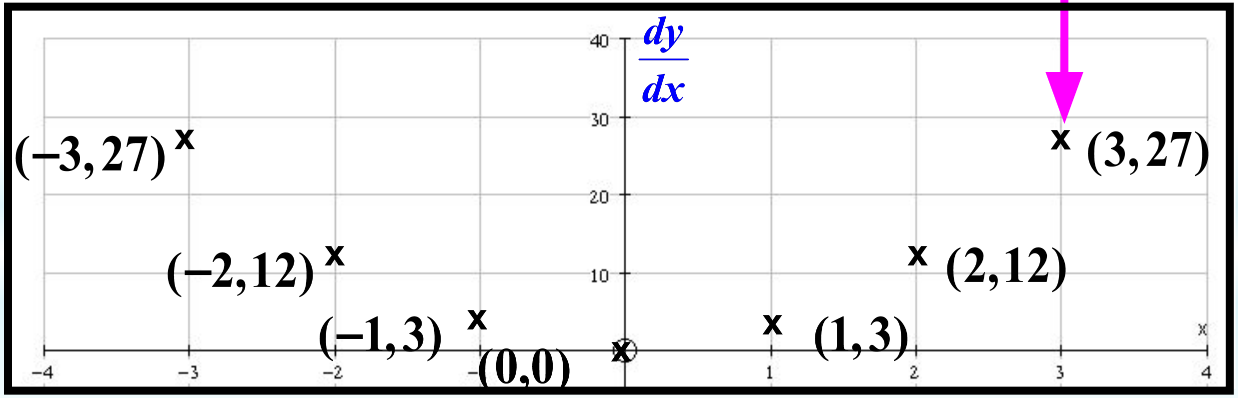

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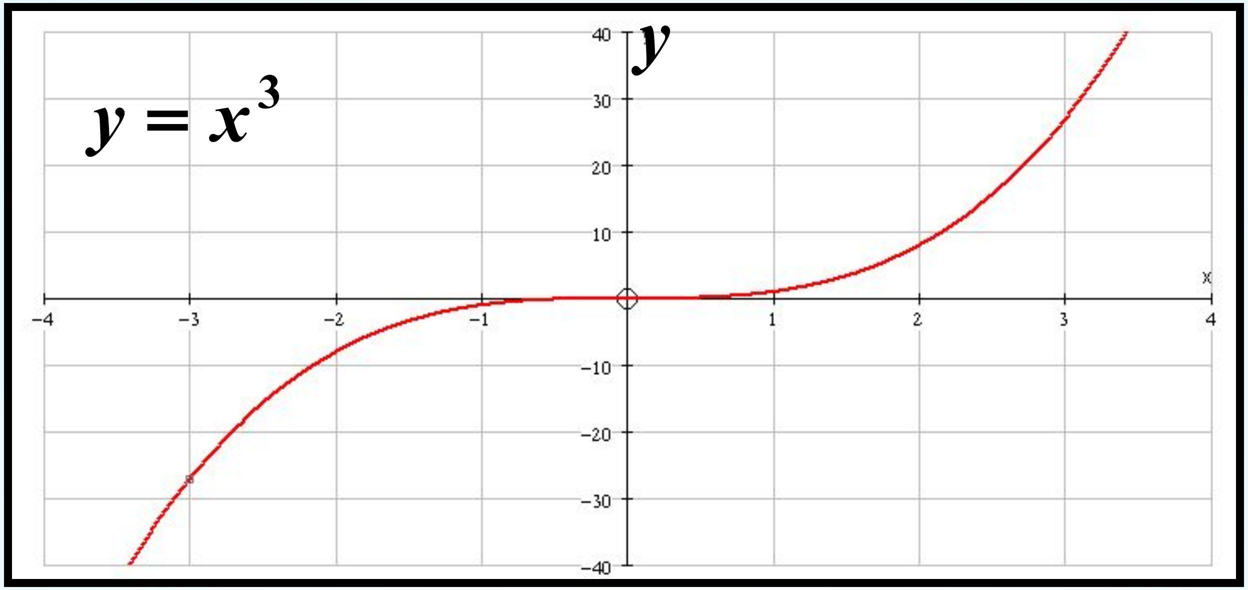
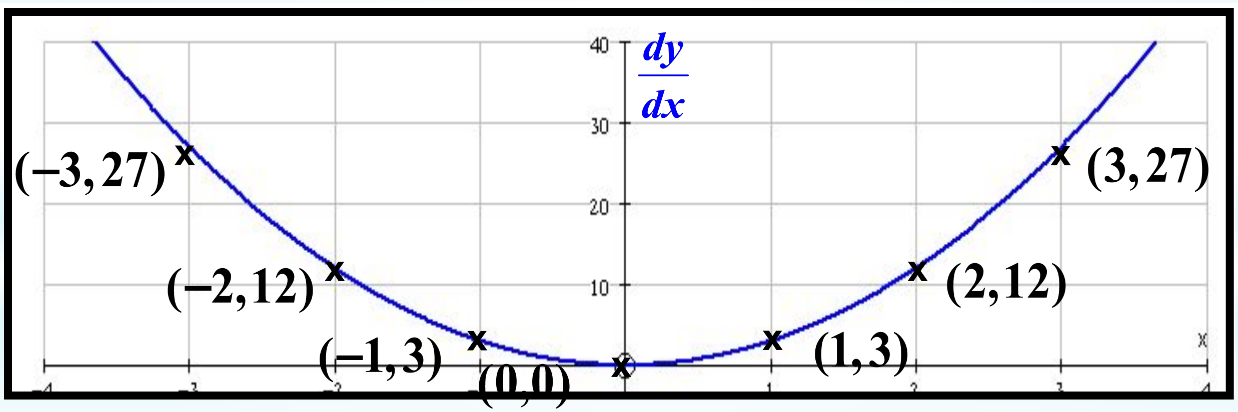
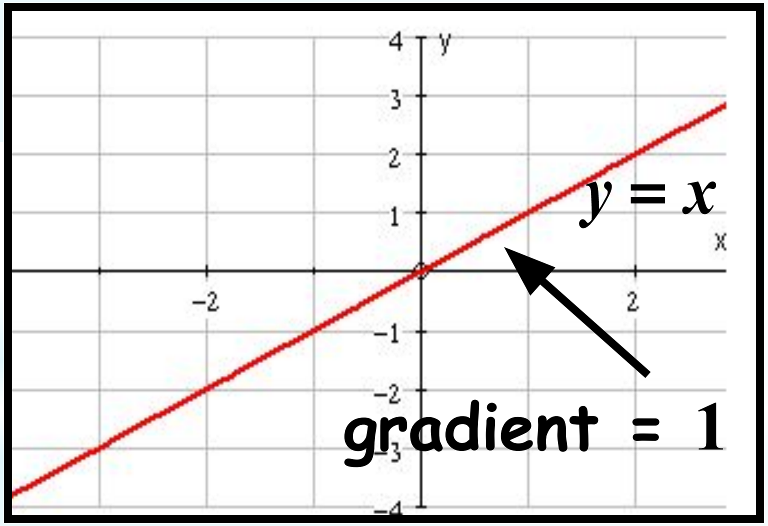
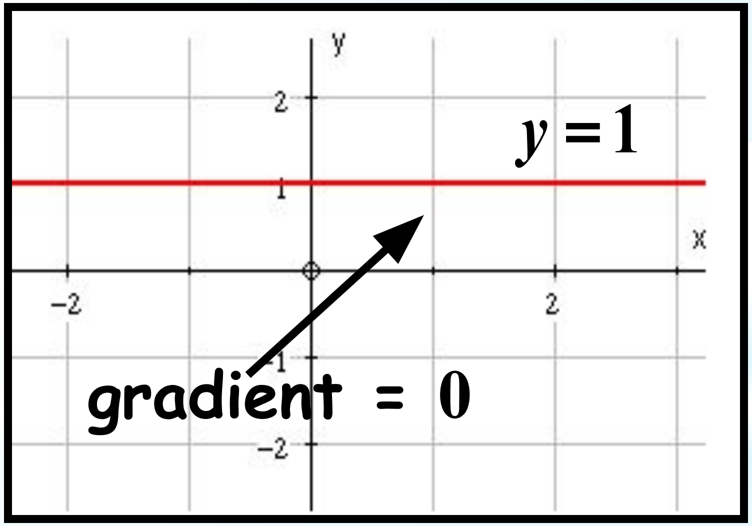




$$\frac{dy}{dx} = 3x^2$$

The gradients of the functions $y = x$ and $y = 1$ can also be found by the rule but as they represent straight lines we already know their gradients





Summary of Gradient Functions:

y	$\frac{dy}{dx}$
1	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$

The Gradient Function and Gradient at a Point

e.g.1 Find the gradient of the curve $y = x^3$ at the point $(2, 12)$.

Solution:

$$y = x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2$$

At $x = 2$, the gradient $m = \frac{dy}{dx} = 3(2)^2$

$$= 12$$

Exercises

1. Find the gradient of the curve $y = x^4$ at the point where $x = -1$

Solution: $\frac{dy}{dx} = 4x^3$ At $x = -1,\ m = -4$

2. Find the gradient of the curve $y = x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$

Solution: $\frac{dy}{dx} = 3x^2$ At $x = \frac{1}{2},\ m = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

- The process of finding the gradient function is called differentiation.
- The gradient function is called the derivative.

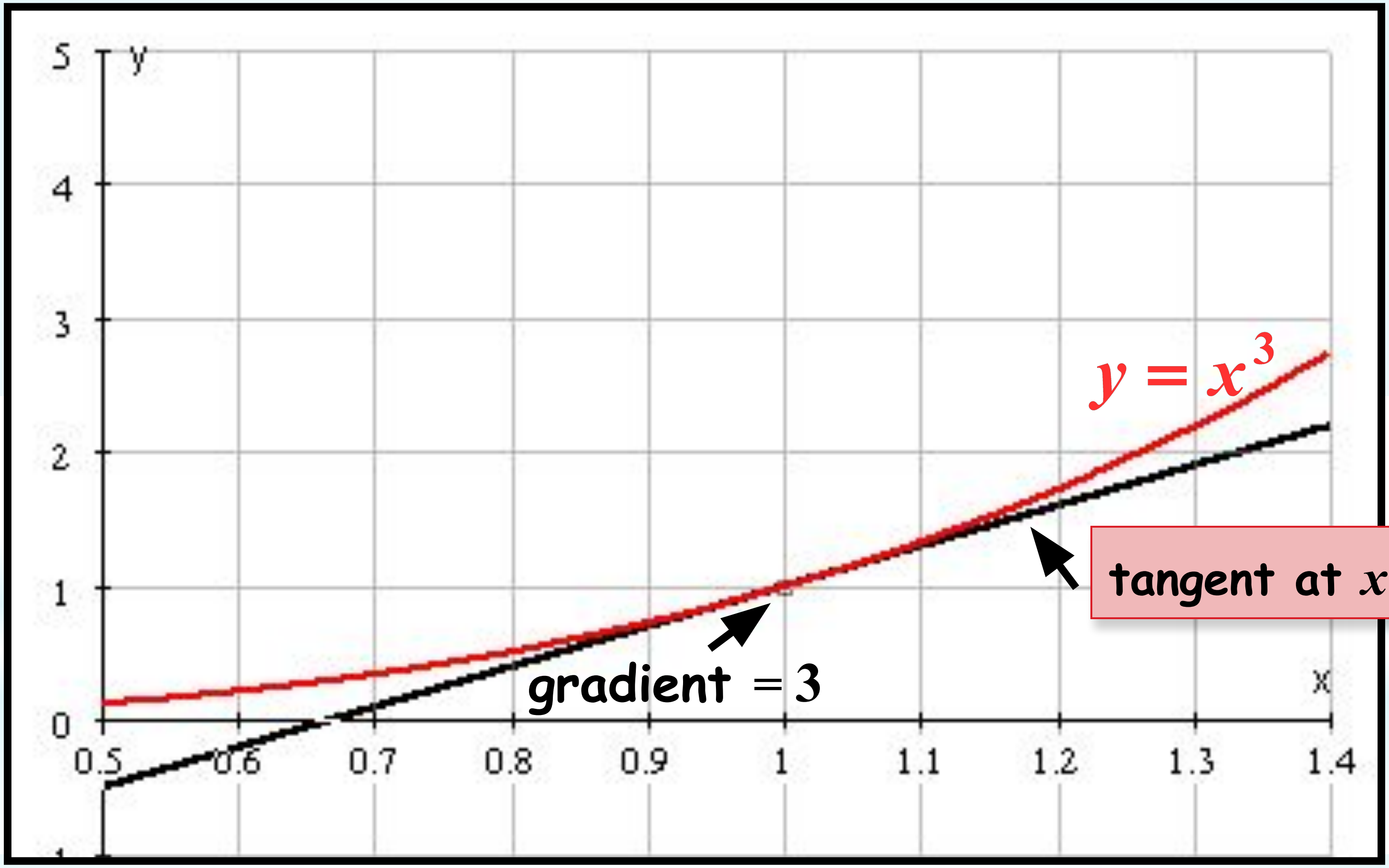
The rule for differentiating can be extended to curves of the form

$$y = ax^n$$

where a is a constant.

More Gradient Functions

e.g. Multiplying $y = x^3$ by 2 multiplies the gradient by 2



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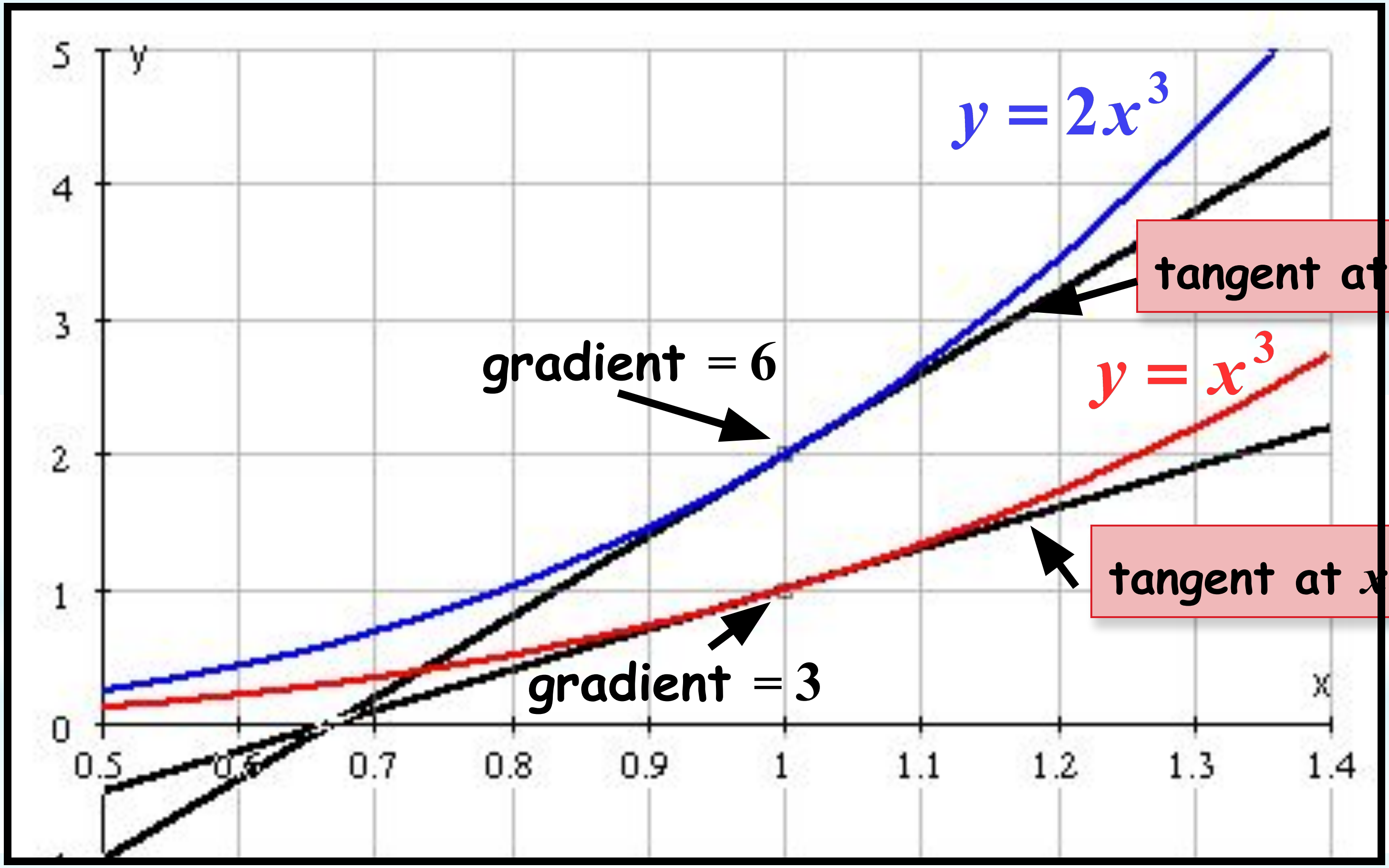

Multiplying x^n by a constant, multiplies the gradient by that constant

e.g. $y = 2x^3 \Rightarrow \frac{dy}{dx} = 2 \times 3x^2 = 6x^2$

The rule can also be used for sums and differences of terms.

For $y = ax^3 + bx^2 + cx + d$ $\quad \frac{dy}{dx} = 3ax^2 + 2bx + c$

For $y = \sum_{i=1}^{n} a_i \cdot x^i + c$ $\quad \frac{dy}{dx} = \sum_{i=1}^{n} a_i \cdot i \cdot x^{i-1}$

e.g. $$y = \frac{1}{2}x^3 - 5x^2 + 7x - 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times 3x^2 - 5 \times 2x + 7 = \frac{3}{2}x^2 - 10x + 7$$

Using Gradient Functions

e.g. Find the gradient at the point where $x = 1$ on the curve

$$y = 3x^3 + 2x^2 - x + 4$$

Solution:

Differentiating to find the gradient function:

$$\frac{dy}{dx} = 9x^2 + 4x - 1$$

When $x = 1$, gradient $m = 9(1)^2 + 4(1) - 1 \Rightarrow m = 12$

SUMMARY

- The gradient at a point on a curve is defined as the gradient of the tangent at that point
- The function that gives the gradient of a curve at any point is called the gradient function
- The process of finding the gradient function is called differentiating
- The rule for differentiating terms of the form

$y = ax^n$ is $\dfrac{dy}{dx} = anx^{n-1}$

- "power to the front and multiply"
- "subtract 1 from the power"

Exercises

Find the gradients at the given points on the following curves:

1. $y = 2x^3 - 5x^2 + 7x - 3$ at the point $(1,1)$

$$\frac{dy}{dx} = 6x^2 - 10x + 7$$ When $x = 1,\quad m = 3$

2. $y = 4x^3 + 3x^2 - 2x + 4$ at the point $(-1,5)$

$$\frac{dy}{dx} = 12x^2 + 6x - 2$$ When $x = -1,\quad m = 4$

3. $y = \frac{1}{2}x^2 - 2x + 1$ at the point $(2,-1)$

$$\frac{dy}{dx} = x - 2$$ When $x = 2,\quad m = 0$

Exercises

Find the gradients at the given points on the following curves:

4. $y=(x-2)(x+4)$ at $(-1,-9)$

(Multiply out the brackets before using the rule)

$y=x^2+2x-8 \Rightarrow \dfrac{dy}{dx}=2x+2$ When $x=-1,\ m=0$

5. $y=\dfrac{x^3-2x}{x}$ at $(2, 2)$

(Divide out before using the rule)

$$y=\frac{x^3}{x}-\frac{2x}{x} \Rightarrow y=x^2-2 \Rightarrow \frac{dy}{dx}=2x$$

When $x=2 \Rightarrow m=4$