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P1 Chapter 6.2

«The Gradient of the Tangent as a Limit»

The Gradient of a Tangent

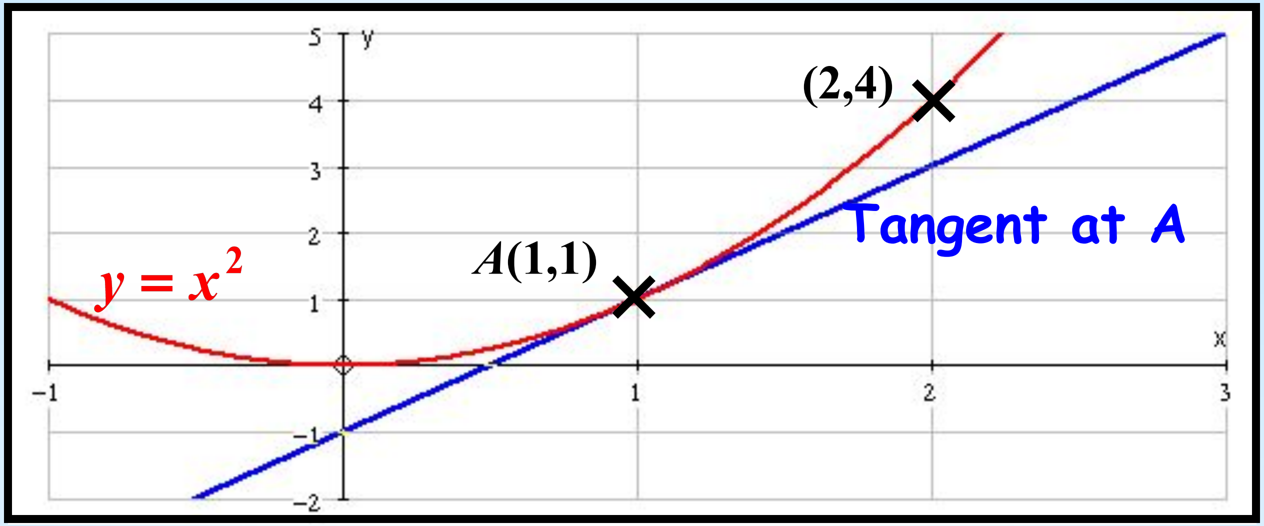
We found the rule for differentiating by noticing a pattern in results found by measuring gradients of tangents.

However, if we want to prove the rule or find a rule for some other functions we need a method based on algebra.

This presentation shows you how this is done.

The emphasis in this presentation is upon understanding ideas rather than doing calculations.

Consider the tangent at the point A(1, 1) on $y = x^2$

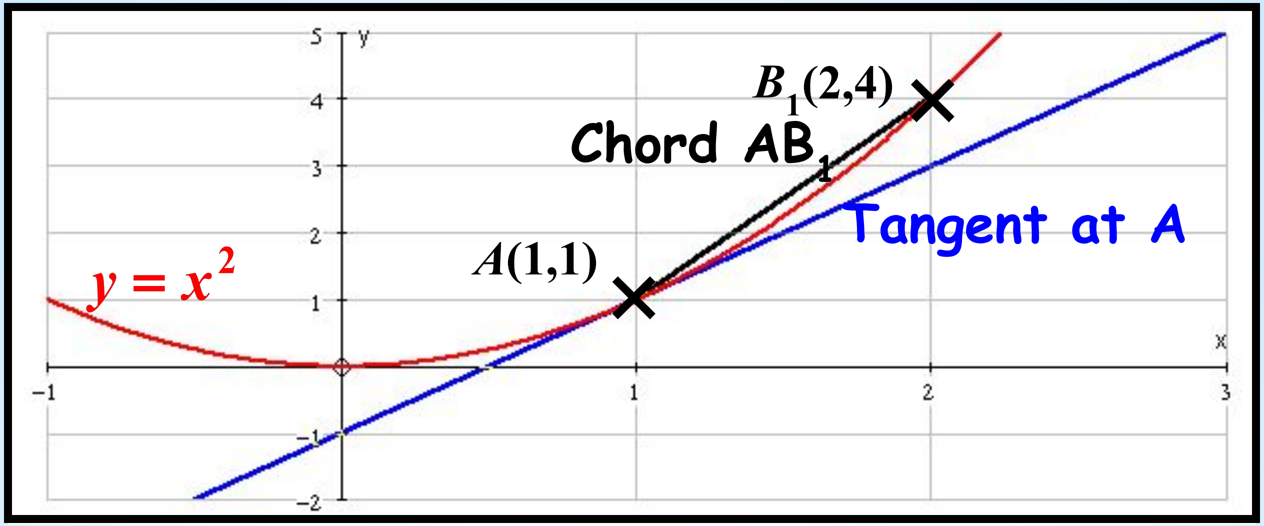


As an approximation to the gradient of the tangent we can use the gradient of a chord from A to a point close to A.

e.g. we can use the chord to the point (2, 4).

(We are going to use several points, so we'll call this point B1).

Consider the tangent at the point $A(1, 1)$ on $y = x^2$

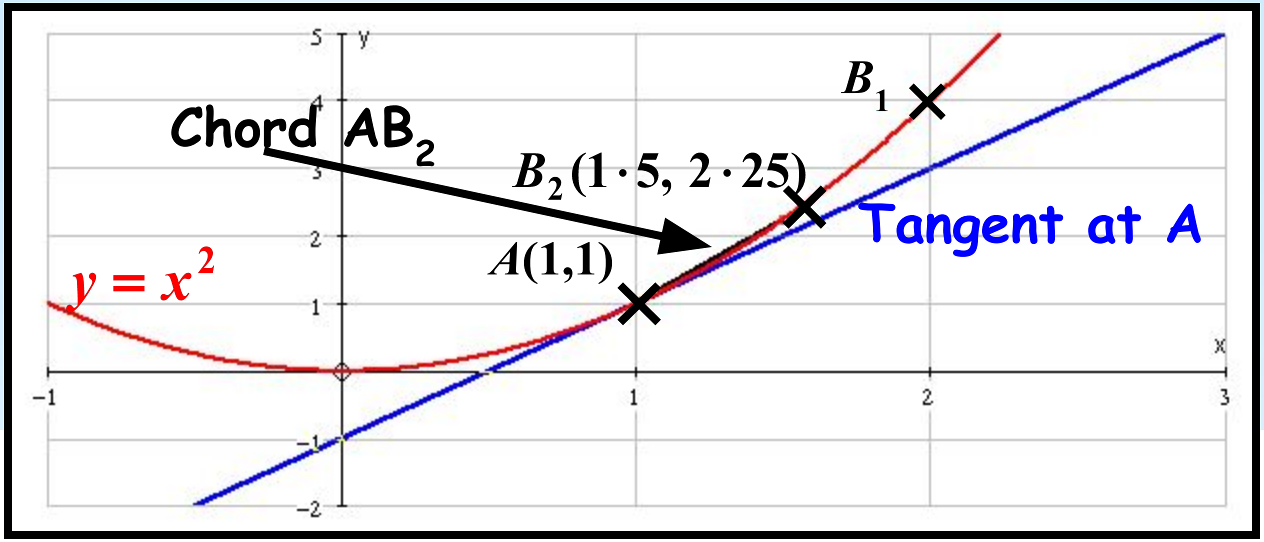


The gradient of the chord AB_1 is given by

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}} \quad \Rightarrow \quad m = \frac{4-1}{2-1} = 3$$

We can see this gradient is larger than the gradient of the tangent.

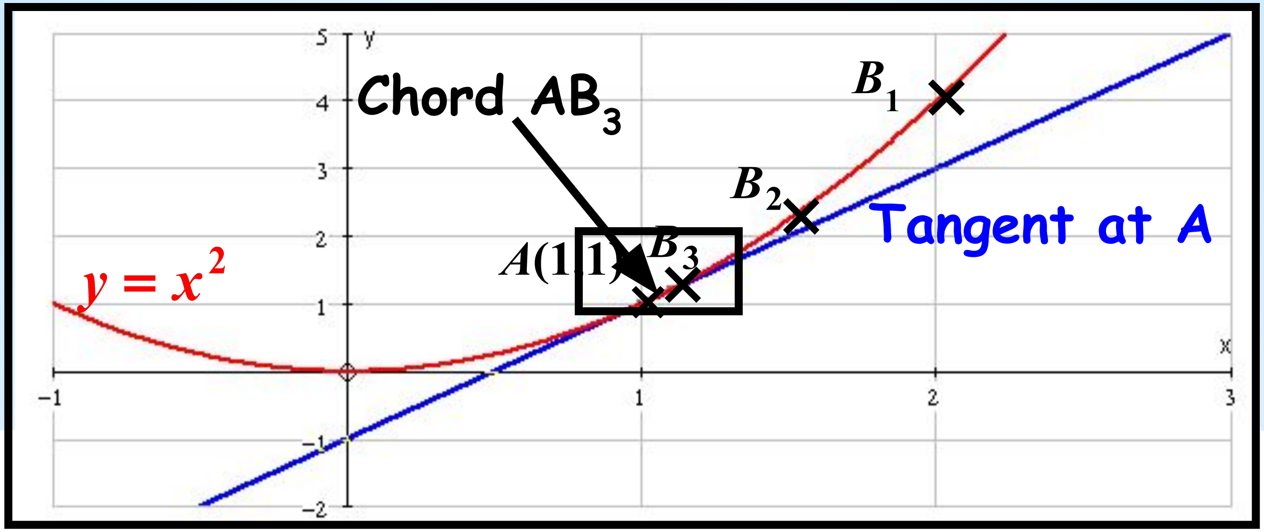
To get a better estimate we can take a point B_2 that is closer to A(1, 1), e.g. $B_2(1\cdot5,\ 2\cdot25)$



The gradient of the chord AB_2 is

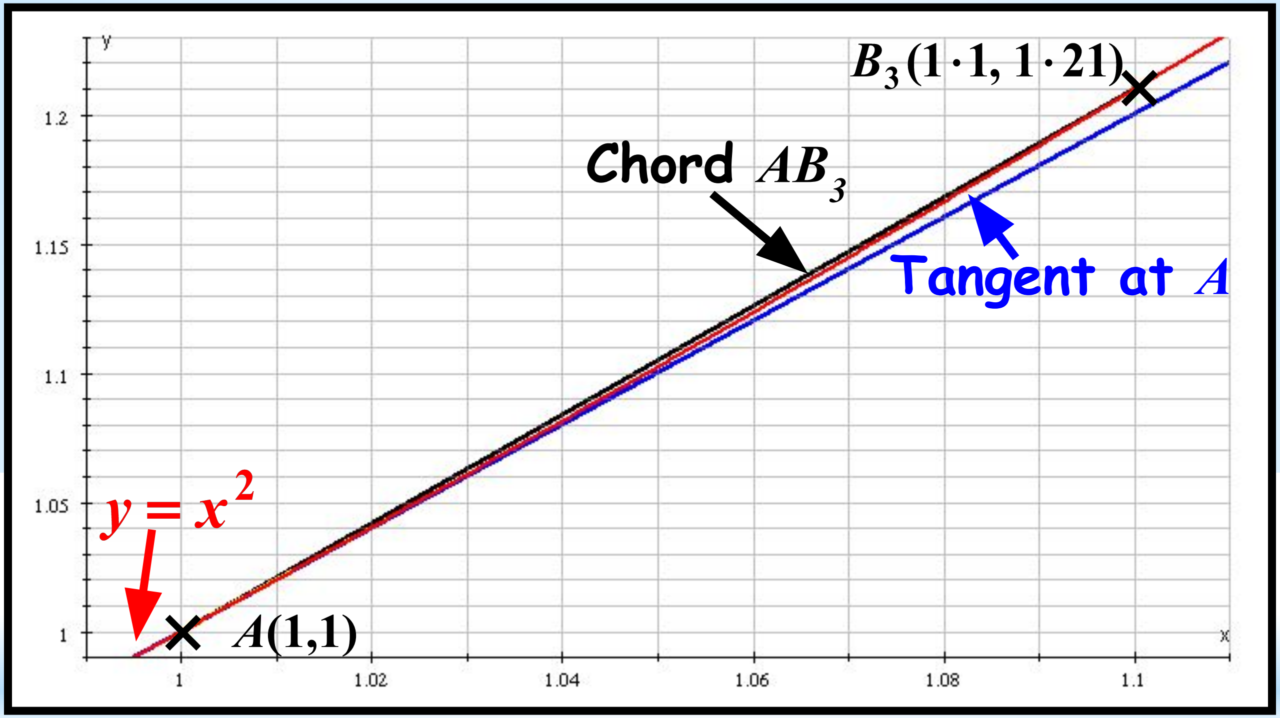
$$m = \frac{2\cdot25 - 1}{1\cdot5 - 1} = 2\cdot5$$

We can get an even better estimate if we use the point $B_3(1 \cdot 1, 1 \cdot 21)$.



We need to zoom in to the curve to see more clearly.

We can get an even better estimate if we use the point $B_3(1\cdot1,\ 1\cdot21)$.



The gradient of AB_3 is $$m = \frac{1\cdot21 - 1}{1\cdot1 - 1} = 2\cdot1$$

Continuing in this way, moving B closer and closer to A(1, 1), and collecting the results in a table, we get

Point	B_1	B_2	B_3	B_4	B_5
x	2	$1\cdot5$	$1\cdot1$	$1\cdot01$	$1\cdot001$
$y\ (=x^2)$	4	$2\cdot25$	$1\cdot21$	$1\cdot0201$	$1\cdot002001$
$y-1$	3	$1\cdot25$	$0\cdot21$	$0\cdot0201$	$0\cdot002001$
$x-1$	1	$0\cdot5$	$0\cdot1$	$0\cdot01$	$0\cdot001$
Gradient of AB	3	$2\cdot5$	$2\cdot1$	$2\cdot01$	$2\cdot001$

As B gets closer to A, the gradient approaches 2.
This is the gradient of the tangent at A.

As B gets closer to A, the gradient of the chord AB approaches the gradient of the tangent.

We write that the gradient of the tangent at A

$$= \lim_{\text{as } B \to A} (\text{ gradient of the chord } AB \text{ })$$

The gradient of the tangent at A is " the limit of the gradient of the chord AB as B approaches A "

We will generalize the result above to find a formula for the gradient at any point on a curve given by

$$y = f(x)$$

Let A be the point $(x, f(x))$ on the curve $y = f(x)$

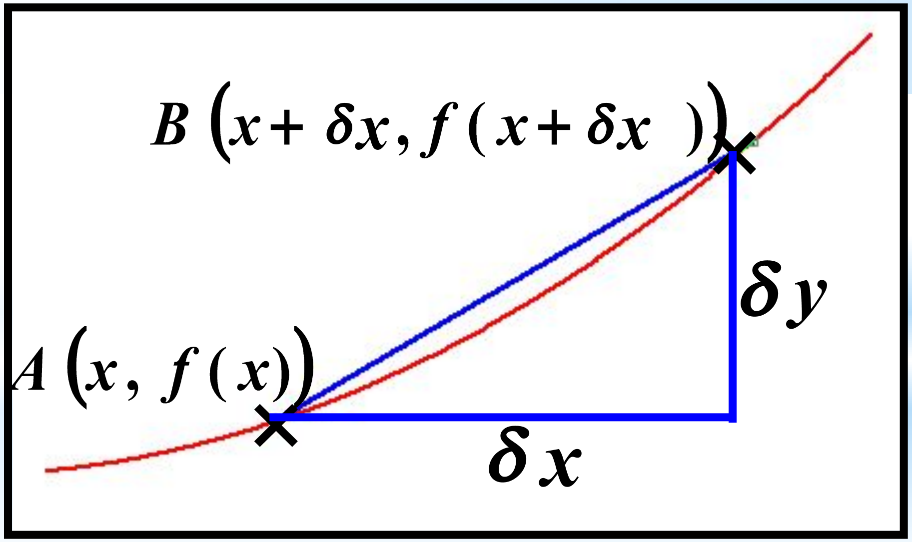
We need a general notation for the coordinates of B that suggests it is near to A.

We use

$$B\left(x+\delta x, f(x+\delta x)\right)$$

δ is the Greek letter d so we can think of δ as standing for "difference".

So, δx is the small difference in x as we move from A to B.



δy can be used for the difference in y values.

We have

$$A\left(x, f(x)\right) \quad \text{and} \quad B\left(x+\delta x, f(x+\delta x)\right)$$

So, the gradient of the chord AB is m where

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}} \quad \Rightarrow \quad m = \frac{f(x+\delta x) - f(x)}{x+\delta x - x}$$

$$\Rightarrow \quad m = \frac{f(x+\delta x) - f(x)}{\delta x}$$

So, since the

gradient of the tangent $= \lim\limits_{\text{as } B \to A}$ (gradient of the chord AB)

$$\boxed{\text{the gradient of the tangent} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}}$$

(We can't put $\delta x = 0$ in this as we would get $\frac{0}{0}$ which is undefined.)

$$\text{the gradient of the tangent} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

(the letter h is sometimes used instead of δx)

If we use δy for the difference in y-values,

$$\delta y = f(x+\delta x) - f(x)$$

We then get

$$\text{the gradient of the tangent} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

But, the gradient of the tangent gives the gradient of the curve, so

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

e.g. $\boxed{\text{Prove}}$ that the gradient function of the curve where $y = f(x)$ is given by $f(x) = x^2$ $\quad \dfrac{dy}{dx} = 2x$

Solution: The gradient of the curve is given by the gradient of the tangent, so

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

For $f(x) = x^2$, $f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2$

$$= x^2 + 2\delta x + \delta x^2 - x^2$$

$$= 2\,\delta x + \delta x^2$$

So, $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{2\delta x + \delta x^2}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{\cancel{\delta x}(2x + \delta x)}{\cancel{\delta x}} = \lim\limits_{\delta x \to 0} (2x + \delta x)$

Since δx has cancelled, we will not be dividing by zero.

So, as $\delta x \to 0$, $\dfrac{dy}{dx} = 2x$