

# Integration of Some Classes of Trigonometric Functions

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# Integration of Some Classes of Trigonometric Functions

In this section we consider 8 forms of integrals with trigonometric functions. Special transformations and substitutions used for each of these classes allow us to obtain exact solutions for these integrals.

1. Integrals of the form  $\int \cos ax \cos bxdx$ ,  $\int \sin ax \cos bxdx$ ,  $\int \sin ax \sin bxdx$ .

To find integrals of this type, use the following trigonometric identities:

- $\cos ax \cos bx = \frac{1}{2} [\cos(ax + bx) + \cos(ax - bx)]$
- $\sin ax \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)]$
- $\sin ax \sin bx = -\frac{1}{2} [\cos(ax + bx) - \cos(ax - bx)]$

For example, 
$$\int \cos(15x) \cos(4x) dx = \frac{1}{2} \int \cos(11x) + \cos(19x) dx$$

$$= \frac{1}{2} \left( \frac{1}{11} \sin(11x) + \frac{1}{19} \sin(19x) \right) + c$$

### Example

Evaluate the integral  $\int \sin \frac{x}{4} \cos \frac{x}{3} dx$ .

*Solution.*

Transform the integrand by the formula

$$\sin ax \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)].$$

Hence,

$$\sin \frac{x}{4} \cos \frac{x}{3} = \frac{1}{2} \left[ \sin \left( \frac{x}{4} + \frac{x}{3} \right) + \sin \left( \frac{x}{4} - \frac{x}{3} \right) \right] = \frac{1}{2} \left( \sin \frac{7x}{12} - \sin \frac{x}{12} \right).$$

Then the integral becomes

$$\int \sin \frac{x}{4} \cos \frac{x}{3} dx = \frac{1}{2} \int \left( \sin \frac{7x}{12} - \sin \frac{x}{12} \right) dx = \frac{1}{2} \left( \frac{\cos \frac{7x}{12}}{\frac{7}{12}} - \frac{\cos \frac{x}{12}}{\frac{1}{12}} \right) + C$$

$$= \frac{6}{7} \cos \frac{7x}{12} - 6 \cos \frac{x}{12} + C.$$

## 2. Integrals of the form $\int \sin^m x \cos^n x dx$

It's assumed here and below that  $m$  and  $n$  are positive integers. To find an integral of this form, use the following substitutions:

- If the power  $n$  of the cosine is odd (the power  $m$  of the sine can be arbitrary), then the substitution  $u = \sin x$  is used.
- If the power  $m$  of the sine is odd, then the substitution  $u = \cos x$  is used.
- If both powers  $m$  and  $n$  are even, then first use the **double angle formulas**

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1,$$

to reduce the power of the sine or cosine in the integrand. Then, if necessary, apply the rules a) or b).

### Example 1

Calculate the integral  $\int \sin^3 x dx$ .

*Solution.*

Let  $u = \cos x$ ,  $du = -\sin x dx$ . Then

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx = \int (1 - \cos^2 x) \sin x dx = -\int (1 - u^2) du = \int (u^2 - 1) du \\ &= \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.\end{aligned}$$

### Example 2

Evaluate the integral  $\int \cos^5 x dx$ .

*Solution.*

Making the substitution  $u = \sin x$ ,  $du = \cos x dx$  and using the identity  $\cos^2 x = 1 - \sin^2 x$ , we obtain

$$\begin{aligned}\int \cos^5 x dx &= \int (\cos^2 x)^2 \cos x dx = \int (1 - \sin^2 x)^2 \cos x dx = \int (1 - u^2)^2 du \\ &= \int (1 - 2u^2 + u^4) du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.\end{aligned}$$

#### Example 4

Calculate the integral  $\int \sin^2 x \cos^4 x dx$ .

*Solution.*

We can write:

$$I = \int \sin^2 x \cos^4 x dx = \int (\sin x \cos x)^2 \cos^2 x dx.$$

Transform the integrand using the identities

$$\sin x \cos x = \frac{\sin 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}.$$

We get

$$\begin{aligned} I &= \int \left( \frac{\sin 2x}{2} \right)^2 \frac{1 + \cos 2x}{2} dx = \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) dx \\ &= \frac{1}{8} \int \sin^2 2x dx + \frac{1}{8} \int \sin^2 2x \cos 2x dx = \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx + \frac{1}{16} \int 2 \sin^2 2x \cos 2x dx \\ &= \frac{1}{16} \int (1 - \cos 4x) dx + \frac{1}{16} \int \sin^2 2x d(\sin 2x) = \frac{1}{16} \left( x - \frac{\sin 4x}{4} \right) + \frac{1}{16} \cdot \frac{\sin^3 2x}{3} + C \\ &= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C. \end{aligned}$$

### 3. Integrals of the form $\int \tan^n x dx$

The power of the integrand can be reduced by using the trigonometric identity  $1 + \tan^2 x = \sec^2 x$  and the **reduction formula**

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

### 4. Integrals of the form $\int \cot^n x dx$

We can reduce the power of the integrand using the trigonometric identity  $1 + \cot^2 x = \csc^2 x$  and the **reduction formula**

$$\int \cot^n x dx = \int \cot^{n-2} x \cot^2 x dx = \int \cot^{n-2} x (\csc^2 x - 1) dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx.$$

### 5. Integrals of the form $\int \sec^n x dx$

This type of integrals can be simplified with help of the reduction formula:

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

### 6. Integrals of the form $\int \csc^n x dx$

Similarly to the previous examples, this type of integrals can be simplified by the formula

$$\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx.$$



**Example 11**

Compute  $\int \tan^3 x \sec^2 x dx$ .

*Solution.*

$$\int \tan^3 x \sec^2 x dx = \int \tan^3 x d(\tan x) = \frac{\tan^4 x}{4} + C.$$

**Example 12**

Compute  $\int \tan^2 x \sec x dx$ .

*Solution.*

Use the identity  $1 + \tan^2 x = \sec^2 x$ . Then

$$I = \int \tan^2 x \sec x dx = \int (\sec^2 x - 1) \sec x dx = \int \sec^3 x dx - \int \sec x dx.$$

Since  $\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right|$  (see Example 9) and  $\int \sec x dx$  is a table integral equal to  $\int \sec x dx = \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right|$ , we obtain the following complete answer:

$$\begin{aligned} I &= \int \sec^3 x dx - \int \sec x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| - \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| + C \\ &= \frac{\sec x \tan x}{2} - \frac{1}{2} \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| + C. \end{aligned}$$

7. Integrals of the form  $\int \tan^m x \sec^n x dx$

- a. If the power of the secant  $n$  is even, then using the identity  $1 + \tan^2 x = \sec^2 x$  the secant function is expressed as the tangent function. The factor  $\sec^2 x$  is separated and used for transformation of the differential. As a result, the entire integral (including differential) is expressed in terms of the function  $\tan x$ .
- b. If both the powers  $n$  and  $m$  are odd, then the factor  $\sec x \tan x$ , which is necessary to transform the differential, is separated. Then the entire integral is expressed in terms of  $\sec x$ .
- c. If the power of the secant  $n$  is odd, and the power of the tangent  $m$  is even, then the tangent is expressed in terms of the secant using the identity  $1 + \tan^2 x = \sec^2 x$ . Then the integrals of the secant are calculated.

8. Integrals of the form  $\int \cot^m x \csc^n x dx$

- a. If the power of the cosecant  $n$  is even, then using the identity  $1 + \cot^2 x = \csc^2 x$  the cosecant function is expressed as the cotangent function. The factor  $\csc^2 x$  is separated and used for transformation of the differential. As a result, the integrand and differential are expressed in terms of through  $\cot x$ .
- b. If both the powers  $n$  and  $m$  are odd, then the factor  $\cot x \csc x$ , which is necessary to transform the differential, is separated. Then the integral is expressed in terms of  $\csc x$ .
- c. If the power of the cosecant  $n$  is odd, and the power of the cotangent  $m$  is even, then the cotangent is expressed in terms of the cosecant using the identity  $1 + \cot^2 x = \csc^2 x$ . Then the integrals of the cosecant are calculated.

### Example 3

Find the integral  $\int \sin^6 x dx$ .

*Solution.*

Using identities  $\sin^2 x = \frac{1 - \cos 2x}{2}$  and  $\cos^2 x = \frac{1 + \cos 2x}{2}$ , we can write:

$$\begin{aligned} I &= \int \sin^6 x dx = \int (\sin^2 x)^3 dx = \frac{1}{8} \int (1 - \cos 2x)^3 dx \\ &= \frac{1}{8} \int (1 - 3 \cos 2x + 3 \cos^2 2x - \cos^3 2x) dx = \frac{x}{8} - \frac{3}{8} \cdot \frac{\sin 2x}{2} + \frac{3}{8} \int \cos^2 2x dx \\ &\quad - \frac{3}{8} \int \cos^3 2x dx. \end{aligned}$$

Calculate the integrals in the latter expression.

$$\int \cos^2 2x dx = \int \frac{1 + \cos 4x}{2} dx = \frac{1}{2} \int (1 + \cos 4x) dx = \frac{1}{2} \left( x + \frac{\sin 4x}{4} \right) = \frac{x}{2} + \frac{\sin 4x}{8}.$$

To find the integral  $\int \cos^3 2x dx$ , we make the substitution  $u = \sin 2x$ ,  $du = 2 \cos 2x dx$ . Then

$$\begin{aligned} \int \cos^3 2x dx &= \frac{1}{2} \int 2 \cos^2 2x \cos 2x dx = \frac{1}{2} \int 2 (1 - \sin^2 2x) \cos 2x dx = \frac{1}{2} \int (1 - u^2) du \\ &= \frac{u}{2} - \frac{u^3}{6} = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}. \end{aligned}$$

Hence, the initial integral is

$$\begin{aligned} I &= \frac{x}{8} - \frac{3 \sin 2x}{16} + \frac{3}{8} \left( \frac{x}{2} + \frac{\sin 4x}{8} \right) - \frac{1}{8} \left( \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} \right) + C = \frac{5x}{16} - \frac{\sin 2x}{4} + \frac{3 \sin 4x}{64} \\ &\quad + \frac{\sin^3 2x}{48} + C. \end{aligned}$$

### Example 7

Evaluate the integral  $\int \tan^4 x dx$ .

*Solution.*

We use the identity  $1 + \tan^2 x = \sec^2 x$  to transform the integral. This yields

$$\begin{aligned}\int \tan^4 x dx &= \int \tan^2 x \tan^2 x dx = \int \tan^2 x (\sec^2 x - 1) dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\ &= \int \tan^2 x d(\tan x) - \int (\sec^2 x - 1) dx = \frac{\tan^3 x}{3} - \tan x + x + C.\end{aligned}$$

### Example 8

Calculate the integral  $\int \cot^5 x dx$ .

*Solution.*

Using the identity  $1 + \cot^2 x = \csc^2 x$ , we have

$$\begin{aligned}\int \cot^5 x dx &= \int \cot^3 x \cot^2 x dx = \int \cot^3 x (\csc^2 x - 1) dx = \int \cot^3 x \csc^2 x dx - \int \cot^3 x dx \\ &= -\int \cot^3 x d(\cot x) - \int \cot x \cot^2 x dx = -\frac{\cot^4 x}{4} - \int \cot x (\csc^2 x - 1) dx \\ &= -\frac{\cot^4 x}{4} - \int \cot x \csc^2 x dx + \int \cot x dx = -\frac{\cot^4 x}{4} - \int \cot x d(\cot x) + \int \frac{d(\sin x)}{\sin x} \\ &= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln|\sin x| + C.\end{aligned}$$

**Example 9**

Calculate the integral  $\int \sec^3 x dx$ .

*Solution.*

We use the reduction formula

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

Hence,

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x dx.$$

The integral  $\int \sec x dx$  is a table integral which is equal to  $\int \sec x dx = \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right|$ . (It can be easily found using the universal trigonometric substitution  $\tan \frac{x}{2}$ .) As a result, the integral becomes

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln \left| \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \right| + C.$$

**Example 10**

Evaluate the integral  $\int \csc^4 x dx$ .

*Solution.*

We use the reduction formula

$$\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx,$$

Hence,

$$\begin{aligned} \int \csc^4 x dx &= -\frac{\csc^2 x \cot x}{3} + \frac{2}{3} \int \csc^2 x dx = -\frac{\csc^2 x \cot x}{3} - \frac{2}{3} \cot x + C \\ &= -\frac{\cot x}{3} (\csc^2 x + 2) + C. \end{aligned}$$

**Example 5**

Evaluate the integral  $\int \sin^3 x \sqrt{\cos x} dx$ .

*Solution.*

Making the substitution  $u = \cos x$ ,  $du = -\sin x dx$  and expressing the sine through cosine with help of the formula  $\sin^2 x = 1 - \cos^2 x$ , we obtain

$$\begin{aligned} \int \sin^3 x \sqrt{\cos x} dx &= \int \sin^2 x \sqrt{\cos x} \sin x dx = \int (1 - \cos^2 x) \sqrt{\cos x} \sin x dx \\ &= - \int (1 - u^2) \sqrt{u} du = - \int \left( u^{\frac{1}{2}} - u^{\frac{5}{2}} \right) du = \frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{7} u^{\frac{7}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C \\ &= \frac{2}{7} \sqrt{\cos^7 x} - \frac{2}{3} \sqrt{\cos^3 x} + C. \end{aligned}$$

**Example 6**

Evaluate the integral  $\int \sin \frac{x}{4} \cos \frac{x}{3} dx$ .

*Solution.*

Transform the integrand by the formula

$$\sin ax \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)].$$

Hence,

$$\sin \frac{x}{4} \cos \frac{x}{3} = \frac{1}{2} \left[ \sin \left( \frac{x}{4} + \frac{x}{3} \right) + \sin \left( \frac{x}{4} - \frac{x}{3} \right) \right] = \frac{1}{2} \left( \sin \frac{7x}{12} - \sin \frac{x}{12} \right).$$

Then the integral becomes

$$\begin{aligned} \int \sin \frac{x}{4} \cos \frac{x}{3} dx &= \frac{1}{2} \int \left( \sin \frac{7x}{12} - \sin \frac{x}{12} \right) dx = \frac{1}{2} \left( \frac{\cos \frac{7x}{12}}{\frac{7}{12}} - \frac{\cos \frac{x}{12}}{\frac{1}{12}} \right) + C \\ &= \frac{6}{7} \cos \frac{7x}{12} - 6 \cos \frac{x}{12} + C. \end{aligned}$$