Integration of Some Classes of Trigonometric Functions

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In this section we consider 8 forms of integrals with trigonometric functions. Special transformations and subtitutions used for each of these classes allow us to obtain exact solutions for these integrals.

1. Integrals of the form $\int \cos ax \cos bx dx$, $\int \sin ax \cos bx dx$, $\int \sin ax \sin bx dx$.

To find integrals of this type, use the following trigonometric identities:

- $\cos ax \cos bx = \frac{1}{2}\left[\cos(ax + bx) + \cos(ax - bx)\right]$
- $\sin ax \cos bx = \frac{1}{2}\left[\sin(ax + bx) + \sin(ax - bx)\right]$
- $\sin ax \sin bx = -\frac{1}{2}\left[\cos(ax + bx) - \cos(ax - bx)\right]$

For example, $\int \cos(15x)\cos(4x)dx = \frac{1}{2}\int \cos(11x)+\cos(19x)\,dx$

$$= \frac{1}{2}\left(\frac{1}{11}\sin(11x)+\frac{1}{19}\sin(19x)\right)+c$$

Example

Evaluate the integral $\int \sin\frac{x}{4}\cos\frac{x}{3}dx$.

Solution.
Transform the integrand by the formula

$$\sin ax\cos bx = \frac{1}{2}[\sin(ax+bx)+\sin(ax-bx)].$$

Hence,

$$\sin\frac{x}{4}\cos\frac{x}{3} = \frac{1}{2}\left[\sin\left(\frac{x}{4}+\frac{x}{3}\right)+\sin\left(\frac{x}{4}-\frac{x}{3}\right)\right] = \frac{1}{2}\left(\sin\frac{7x}{12}-\sin\frac{x}{12}\right).$$

Then the integral becomes

$$\int \sin\frac{x}{4}\cos\frac{x}{3}dx = \frac{1}{2}\int\left(\sin\frac{7x}{12}-\sin\frac{x}{12}\right)dx = \frac{1}{2}\left(\frac{\cos\frac{7x}{12}}{\frac{7}{12}}-\frac{\cos\frac{x}{12}}{\frac{1}{12}}\right)+C$$

$$= \frac{6}{7}\cos\frac{7x}{12}-6\cos\frac{x}{12}+C.$$

2. *Integrals of the form* $\int \sin^m x \cos^n x dx$

It's assumed here and below that m and n are positive integers. To find an integral of this form, use the following substitutions:

a. If the power n of the cosine is odd (the power m of the sine can be arbitrary), then the substitution $u = \sin x$ is used.

b. If the power m of the sine is odd, then the substitution $u = \cos x$ is used.

c. If both powers m and n are even, then first use the double angle formulas

$$\sin 2x = 2 \sin x \cos x, \;\; \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1,$$

to reduce the power of the sine or cosine in the integrand. Then, if necessary, apply the rules a) or b).

Example 1

Calculate the integral $\int \sin^3 x dx$.

Solution.
Let $u = \cos x$, $du = -\sin x dx$. Then

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int \left(1 - \cos^2 x\right) \sin x dx = -\int \left(1 - u^2\right) du = \int \left(u^2 - 1\right) du$$
$$= \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

Example 2

Evaluate the integral $\int \cos^5 x dx$.

Solution.
Making the substitution $u = \sin x$, $du = \cos x dx$ and using the identity $\cos^2 x = 1 - \sin^2 x$, we obtain

$$\int \cos^5 x dx = \int \left(\cos^2 x\right)^2 \cos x dx = \int \left(1 - \sin^2 x\right)^2 \cos x dx = \int \left(1 - u^2\right)^2 du$$
$$= \int \left(1 - 2u^2 + u^4\right) du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$$

Example 4

Calculate the integral $\int \sin^2 x \cos^4 x dx$.

Solution.
We can write:

$$I = \int \sin^2 x \cos^4 x dx = \int (\sin x \cos x)^2 \cos^2 x dx.$$

Transform the integrand using the identities

$$\sin x \cos x = \frac{\sin 2x}{2},\ \cos^2 x = \frac{1+\cos 2x}{2},\ \sin^2 x = \frac{1-\cos 2x}{2}.$$

We get

$$\begin{aligned} I &= \int \left(\frac{\sin 2x}{2}\right)^2 \frac{1+\cos 2x}{2} dx = \frac{1}{8}\int \sin^2 2x\,(1+\cos 2x)\,dx \\ &= \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cos 2x dx = \frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{16}\int 2\sin^2 2x \cos 2x dx \\ &= \frac{1}{16}\int (1-\cos 4x)\,dx + \frac{1}{16}\int \sin^2 2x\, d(\sin 2x) = \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \frac{1}{16}\cdot\frac{\sin^3 2x}{3} + C \\ &= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C. \end{aligned}$$

3. Integrals of the form $\int \tan^n x dx$

The power of the integrand can be reduced by using the trigonometric identity $1 + \tan^2 x = \sec^2 x$ and the reduction formula

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

4. Integrals of the form $\int \cot^n x dx$

We can reduce the power of the integrand using the trigonometric identity $1 + \cot^n x = \csc^n x$ and the reduction formula

$$\int \cot^n x dx = \int \cot^{n-2} x \cot^2 x dx = \int \cot^{n-2} x \left(\csc^2 x - 1\right) dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx.$$

5. Integrals of the form $\int \sec^n x dx$

This type of integrals can be simplified with help of the reduction formula:

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

6. Integrals of the form $\int \csc^n x dx$

Similarly to the previous examples, this type of integrals can be simplified by the formula

$$\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx.$$

Example 11

Compute $\int \tan^3 x \sec^2 x dx$.

Solution.

$$\int \tan^3 x \sec^2 x dx = \int \tan^3 x \, d(\tan x) = \frac{\tan^4 x}{4} + C.$$

Example 12

Compute $\int \tan^2 x \sec x dx$.

Solution.
Use the identity $1 + \tan^2 x = \sec^2 x$. Then

$$I = \int \tan^2 x \sec x dx = \int (\sec^2 x - 1) \sec x dx = \int \sec^3 x dx - \int \sec x dx.$$

Since $\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right|$ (see Example 9) and $\int \sec x dx$ is a table integral equal to $\int \sec x dx = \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right|$, we obtain the following complete answer:

$$I = \int \sec^3 x dx - \int \sec x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| - \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C$$
$$= \frac{\sec x \tan x}{2} - \frac{1}{2} \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C.$$

7. *Integrals of the form* $\int \tan^m x \sec^n x dx$

a. If the power of the secant n is even, then using the identity $1 + \tan^2 x = \sec^2 x$ the secant function is expressed as the tangent function. The factor $\sec^2 x$ is separated and used for transformation of the differential. As a result, the entire integral (including differential) is expressed in terms of the function $\tan x$.

b. If both the powers n and m are odd, then the factor $\sec x \tan x$, which is necessary to transform the differential, is separated. Then the entire integral is expressed in terms of $\sec x$.

c. If the power of the secant n is odd, and the power of the tangent m is even, then the tangent is expressed in terms of the secant using the identity $1 + \tan^2 x = \sec^2 x$. Then the integrals of the secant are calculated.

8. *Integrals of the form* $\int \cot^m x \csc^n x dx$

a. If the power of the cosecant n is even, then using the identity $1 + \cot^2 x = \csc^2 x$ the cosecant function is expressed as the cotangent function. The factor $\csc^2 x$ is separated and used for transformation of the differential. As a result, the integrand and differential are expressed in terms of through $\cot x$.

b. If both the powers n and m are odd, then the factor $\cot x \csc x$, which is necessary to transform the differential, is separated. Then the integral is expressed in terms of $\csc x$.

c. If the power of the cosecant n is odd, and the power of the cotangent m is even, then the cotangent is expressed in terms of the cosecant using the identity $1 + \cot^2 x = \csc^2 x$. Then the integrals of the cosecant are calculated.

Example 3

Find the integral $\int \sin^6 x dx$.

Solution.
Using identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$, we can write:

$$I = \int \sin^6 x dx = \int \left(\sin^2 x\right)^3 dx = \frac{1}{8}\int (1 - \cos 2x)^3 dx$$
$$= \frac{1}{8}\int \left(1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\right) dx = \frac{x}{8} - \frac{3}{8}\cdot\frac{\sin 2x}{2} + \frac{3}{8}\int \cos^2 2x dx$$
$$- \frac{3}{8}\int \cos^3 2x dx.$$

Calculate the integrals in the latter expression.

$$\int \cos^2 2x dx = \int \frac{1+\cos 4x}{2} dx = \frac{1}{2}\int (1 + \cos 4x)\, dx = \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) = \frac{x}{2} + \frac{\sin 4x}{8}.$$

To find the integral $\int \cos^3 2x dx$, we make the substitution $u = \sin 2x$, $du = 2\cos 2x dx$. Then

$$\int \cos^3 2x dx = \frac{1}{2}\int 2\cos^2 2x \cos 2x dx = \frac{1}{2}\int 2\left(1 - \sin^2 2x\right)\cos 2x dx = \frac{1}{2}\int \left(1 - u^2\right) du$$
$$= \frac{u}{2} - \frac{u^3}{6} = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}.$$

Hence, the initial integral is

$$I = \frac{x}{8} - \frac{3\sin 2x}{16} + \frac{3}{8}\left(\frac{x}{2} + \frac{\sin 4x}{8}\right) - \frac{1}{8}\left(\frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}\right) + C = \frac{5x}{16} - \frac{\sin 2x}{4} + \frac{3\sin 4x}{64}$$
$$+ \frac{\sin^3 2x}{48} + C.$$

Example 7

Evaluate the integral $\int \tan^4 x dx$.

Solution.
We use the identity $1 + \tan^2 x = \sec^2 x$. to transform the integral. This yields

$$\int \tan^4 x dx = \int \tan^2 x \tan^2 x dx = \int \tan^2 x \left(\sec^2 x - 1\right) dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$$= \int \tan^2 x \, d\left(\tan x\right) - \int \left(\sec^2 x - 1\right) dx = \frac{\tan^3 x}{3} - \tan x + x + C.$$

Example 8

Calculate the integral $\int \cot^5 x dx$.

Solution.
Using the identity $1 + \cot^2 x = \csc^2 x$, we have

$$\int \cot^5 x dx = \int \cot^3 x \cot^2 x dx = \int \cot^3 x \left(\csc^2 x - 1\right) dx = \int \cot^3 x \csc^2 x dx - \int \cot^3 x dx$$

$$= -\int \cot^3 x \, d\left(\cot x\right) - \int \cot x \cot^2 x dx = -\frac{\cot^4 x}{4} - \int \cot x \left(\csc^2 x - 1\right) dx$$

$$= -\frac{\cot^4 x}{4} - \int \cot x \csc^2 x dx + \int \cot x dx = -\frac{\cot^4 x}{4} - \int \cot x \, d\left(\cot x\right) + \int \frac{d\left(\sin x\right)}{\sin x}$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln\left|\sin x\right| + C.$$

Example 9

Calculate the integral $\int \sec^3 x dx$.

Solution.
We use the reduction formula

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

Hence,

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x dx.$$

The integral $\int \sec x dx$ is a table integral which is equal to $\int \sec x dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$. (It can be easily found using the universal trigonometric substitution $\tan \frac{x}{2}$.) As a result, the integral becomes

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C.$$

Example 10

Evaluate the integral $\int \csc^4 x dx$.

Solution.
We use the reduction formula

$$\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx,$$

Hence,

$$\int \csc^4 x dx = -\frac{\csc^2 x \cot x}{3} + \frac{2}{3} \int \csc^2 x dx = -\frac{\csc^2 x \cot x}{3} - \frac{2}{3} \cot x + C$$
$$= -\frac{\cot x}{3} \left(\csc^2 x + 2\right) + C.$$

Example 5

Evaluate the integral $\int \sin^3 x \sqrt{\cos x}\, dx$.

Solution.
Making the substitution $u = \cos x$, $du = -\sin x\, dx$ and expressing the sine through cosine with help of the formula $\sin^2 x = 1 - \cos^2 x$, we obtain

$$\int \sin^3 x \sqrt{\cos x}\, dx = \int \sin^2 x \sqrt{\cos x} \sin x\, dx = \int \left(1 - \cos^2 x\right) \sqrt{\cos x} \sin x\, dx$$
$$= -\int \left(1 - u^2\right) \sqrt{u} du = -\int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}}\right) du = \frac{u^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{7} u^{\frac{7}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C$$
$$= \frac{2}{7} \sqrt{\cos^7 x} - \frac{2}{3} \sqrt{\cos^3 x} + C.$$

Example 6

Evaluate the integral $\int \sin \frac{x}{4} \cos \frac{x}{3} dx$.

Solution.
Transform the integrand by the formula

$$\sin ax \cos bx = \frac{1}{2} [\sin(ax + bx) + \sin(ax - bx)] .$$

Hence,

$$\sin \frac{x}{4} \cos \frac{x}{3} = \frac{1}{2} \left[\sin\left(\frac{x}{4} + \frac{x}{3}\right) + \sin\left(\frac{x}{4} - \frac{x}{3}\right)\right] = \frac{1}{2} \left(\sin \frac{7x}{12} - \sin \frac{x}{12}\right).$$

Then the integral becomes

$$\int \sin \frac{x}{4} \cos \frac{x}{3} dx = \frac{1}{2} \int \left(\sin \frac{7x}{12} - \sin \frac{x}{12}\right) dx = \frac{1}{2} \left(\frac{\cos \frac{7x}{12}}{\frac{7}{12}} - \frac{\cos \frac{x}{12}}{\frac{1}{12}}\right) + C$$
$$= \frac{6}{7} \cos \frac{7x}{12} - 6 \cos \frac{x}{12} + C.$$