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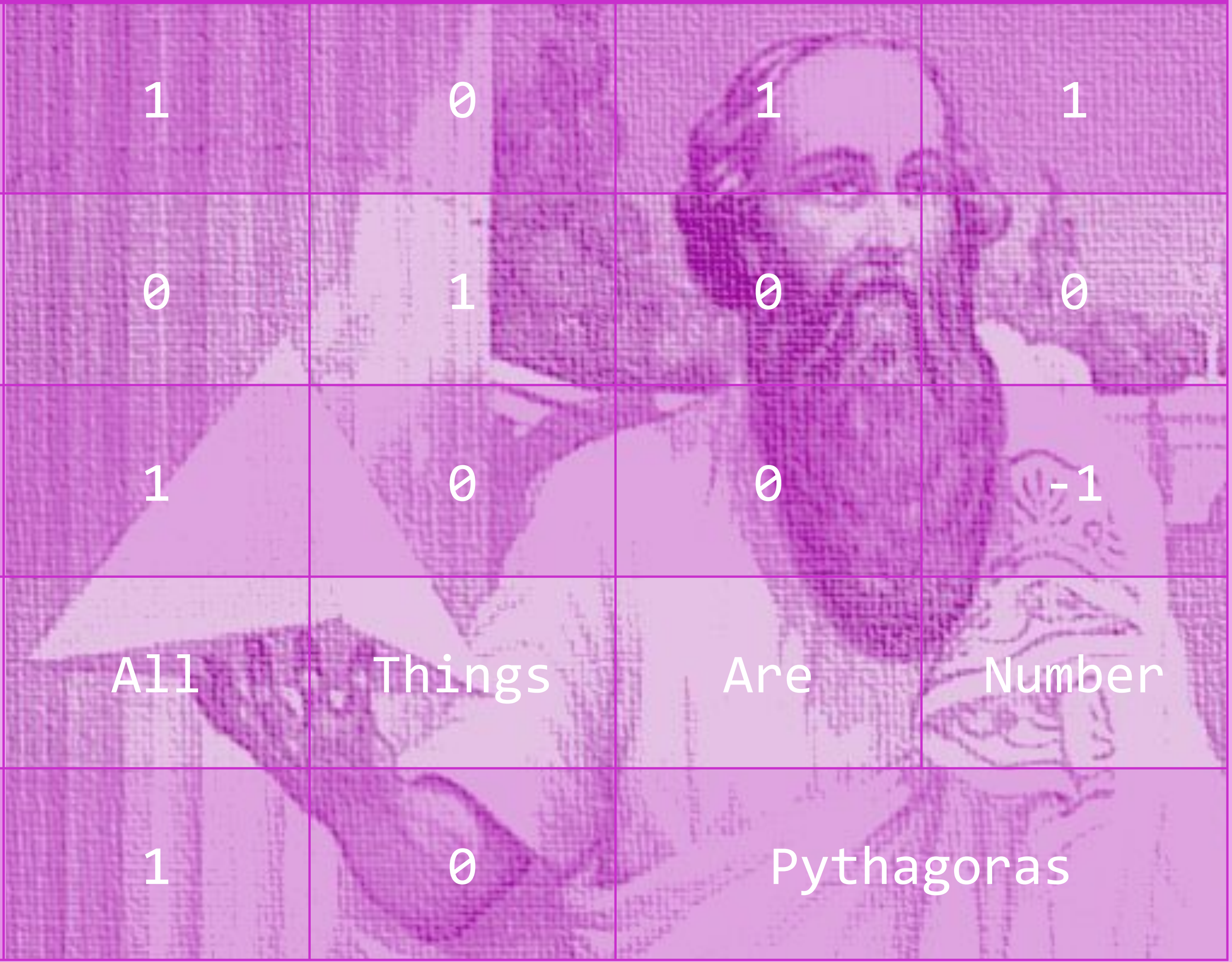


Quan	1	0	1	1
Tum	0	1	0	0
Se	1	0	0	-1
Man	All	Things	Are	Number
Tics	1	0	Pythagoras	

Bra-ket notation (Dirac, 1939)

- V – vector (Hilbert) space, $\mathbb{F}$ – field
- $|\psi\rangle$ – *pure state* (vector, or operator $\mathbb{F} \to V$) – ket
- $\langle\psi|$ – *effect of state* $|\psi\rangle$ (dual vector, dual operator $V \to \mathbb{F}$ Hermitian conjugate) – bra
- *Inner product* of $|\psi\rangle$ and $|\varphi\rangle$ is $\langle\psi|\varphi\rangle$

Outer product

- *Outer product* $|w\rangle\langle v|$ for $w \in W, v \in V$ is a $V \to W$ operator:

$$(|w\rangle\langle v|)(|v'\rangle) \equiv |w\rangle\langle v|v'\rangle = \langle v|v'\rangle|w\rangle$$

- Arbitrary $A: V \to W$ can be written

  in a basis $\{|v_i\rangle\}_i$ for V and $\{|w_j\rangle\}_j$ for W:

$$A = \sum_{ij} m_{ij} |w_j\rangle\langle v_i|,$$

where $m_{ij} = \langle w_j|A|v_i\rangle$

Eigenvectors and eigenvalues

- $A = \sum_i \lambda_i |i\rangle\langle i|$, if $|i\rangle$ is an orthogonal basis in which A is diagonal.
- $|i\rangle$ – are *eigen vectors*
- λ_i – are *eigen values*
- Easy to check:

$$A|i\rangle = \lambda_i |i\rangle$$

Density operator (matrix)

- If $|\varphi_i\rangle$ – are pure states,
- $\{p_i\}$ – are probabilities over them, then

$\rho \equiv \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ is a *dense operator*

- *Positive operator*:

$\langle v|A|v\rangle \geq 0$ for all v

- **Theorem**: ρ is a density operator iff it's a positive Hermitian operator with trace $=1$.

Trace inner product

- A and B are density matrices same dimension
- $tr(A^T B)$
- $A = \sum_i p_i |i\rangle\langle i|$ and $B = \sum_j q_j |j\rangle\langle j|$
- $tr(A^T B) = tr\left(\sum_i p_i |i\rangle\langle i| \sum_j q_j |j\rangle\langle j|\right) =$

$$= \sum_{ij} p_i q_j \langle i|j\rangle\, tr(|i\rangle\langle j|)$$

$$= \sum_{ij} p_i q_j \langle i|j\rangle \langle i|j\rangle$$

$$= \sum_{ij} p_i q_j \langle i|j\rangle^2$$

Distributional Semantics

- "You shall know the word by the company it keeps" (Firth)
  - Obtain meaning high dimensional vector representations from large corpora automatically
- Compositionality
  - DS can not be applied for entire sentence (lack of frequency)
- Entailment
  - w entails v if the meaning of a word w is included in the meaning of a word v (w *is–a* v) – *subsumption relation*
    - non symmetric

Distributional Inclusion Hypothesis

- If u is *semantically narrower* than v, then a significant number of salient distributional features of u are also included in the feature vector of v :
  - **Hypothesis 1**: If $v => w$ then all the characteristic features of v is expected to appear in w.
  - **Hypothesis 2**: If all the characteristic features of v appear in w, then $v => w$.

Category Theory

- A monoidal category C is a category consisting of the following:
  - a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*
  - an object $I \in \mathcal{C}$ called the *unit object*
  - a natural isomorphism whose components

    $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$ are called the *associators*
  - a natural isomorphism whose components $I \otimes A \xrightarrow{\lambda_A} A$ are called the *left unitors*
  - a natural isomorphism whose components $A \otimes I \xrightarrow{\rho_A} A$ are called the *right unitors*

Category Theory

- The objects of the category are thought to be <u>types</u> of systems
- A morphism $f : A \to B$ is a process that takes a system of type A to a system of type B.
- for $f : A \to B$ and $g : B \to C$, $g \circ f$ is the composite morphism that takes a system of type A into a system of type C by applying the process g after f.
- Morphisms of type $\psi : I \to A$ are called elements of A.

Compact closed categories

- A monoidal category is *compact closed* if for each object A, there are also left and right dual objects A^r and A^l, and morphisms

$$\eta^l : I \to A \otimes A^l \quad \eta^r : I \to A \otimes A^r$$
$$\epsilon^l : A^l \otimes A \to I \quad \epsilon^r : A \otimes A^r \to I$$

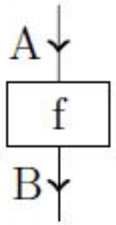
- that satisfies

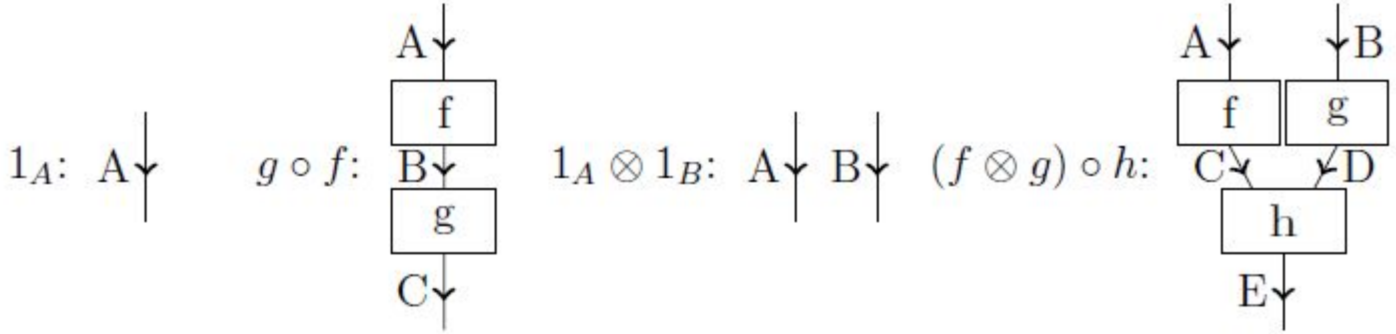
$$\left(1_A \otimes \epsilon^l\right) \circ \left(\eta^l \otimes 1_A\right) = 1_A$$
$$(\epsilon^r \otimes 1_A) \circ (1_A \otimes \eta^r) = 1_A$$
$$\left(\epsilon^l \otimes 1_{A^l}\right) \circ \left(1_{A^l} \otimes \eta^l\right) = 1_{A^l}$$
$$(1_{A^r} \otimes \epsilon^r) \circ (\eta^r \otimes 1_{A^r}) = 1_{A^r}$$

- The maps of compact categories are used to represent *correlations*, and in categorical quantum mechanics they model maximally entangled states.

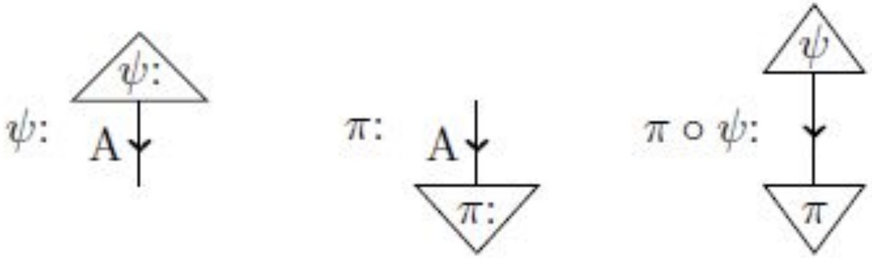
Graphical calculus

$f: A \to B$



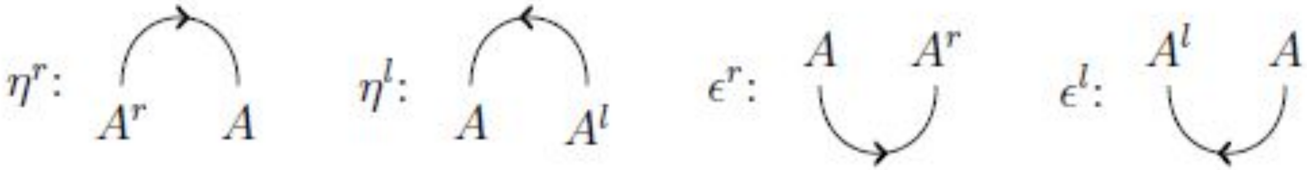


$\psi: I \to A$



$\pi: A \to I$

Graphical calculus



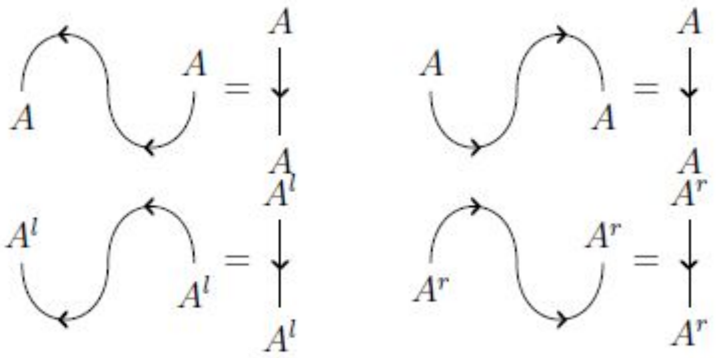
Snake identities

$(1_A \otimes \epsilon^l) \circ (\eta^l \otimes 1_A) = 1_A$

$(\epsilon^r \otimes 1_A) \circ (1_A \otimes \eta^r) = 1_A$

$(\epsilon^l \otimes 1_{A^l}) \circ (1_{A^l} \otimes \eta^l) = 1_{A^l}$

$(1_{A^r} \otimes \epsilon^r) \circ (\eta^r \otimes 1_{A^r}) = 1_{A^r}$



Swing rule

Compositional Distributional Model

- Pregroup grammars (Lambek)
- A partially ordered monoid $(P, \leq, \cdot, 1)$ consists of:
  - a set P
  - a monoid multiplication operator $\cdot : P \times P \to \mathrm{P}$ satisfying the condition
    $$(a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for all } a, b, c \in P$$
  - and thy monoidal unit $1 \in P$ where for all $a \in P$
    $$a \cdot 1 = a = 1 \cdot a$$
  - a partial order $\leq$ on P

Pregroup (Lambek, 2001)

- A *pregroup* $(P, \leq, \cdot, 1, (-)^l, (-)^r)$ is a partially ordered monoid in which each element a has both a left adjoint a^l and a right adjoint a^r such that

$$a^l a \leq 1 \leq a a^l \text{ and } a a^r \leq 1 \leq a^r a$$

- Adjoints have properties:
  - Uniqueness: Adjoints are unique
  - Order reversal: If $a \leq b$ then $b^r \leq a^r$ and $b^l \leq a^l$
  - The unit is self adjoint: $1^l = 1 = 1^r$
  - Multiplication operation is self adjoint: $(a \cdot b)^l = b^l \cdot a^l$ and $(a \cdot b)^r = b^r \cdot a^r$
  - Opposite adjoints annihilate: $(a^r)^l = a = \left(a^l\right)^r$
  - Same adjoints iterate: $a^{ll} a^l \leq 1 \leq a^{rr} a^r, a^{lll} a^{ll} \leq 1 \leq a^{rrr} a^{rr}, \ldots$

Pregroup grammar

- $a \to b$ means $a \leq b$ (a *reduces* to b)
- "John likes Mary"
- "John" and "Mary" assigned to type n (noun)
- "likes" is assigned to compound type $(n^r s n^l)$
- "likes" takes a noun from the left and from the right, and returns a sentence

$$n(n^r s n^l)n \to 1sn^l n \to 1s1 \to s$$

Basic types

- n : noun
- j : infinitive of the verb

s : declarative statement

g: glueing type

Pregroups as compact closed categories

- P is a concrete instance of a compact closed category

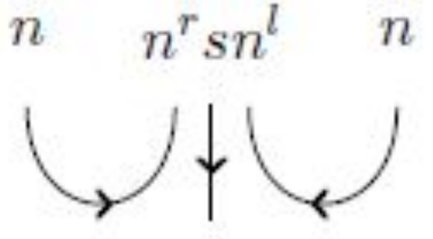
$$\eta^l = [1 \leq p \cdot p^l] \qquad \epsilon^l = [p^l \cdot p \leq 1]$$
$$\eta^r = [1 \leq p^r \cdot p] \qquad \epsilon^r = [p \cdot p^r \leq 1]$$

- Test snake identities:

$$\left(1^p \otimes \epsilon_p^l\right) \circ \left(\eta_p^l \otimes 1_p\right):$$
$$p = 1p \leq pp^l p \leq p1 = p$$
$$\dots$$

Examples

- “John likes Mary”



- “John does not like Mary”

FVect - finite dimensional vector space

- **FVect** – finite dimensional vector spaces over the base field R together with linear maps, form a monoidal category
- FVect as a compact closed category.

Given a vector space V with basis $\{\overrightarrow{e_i}\}_i$

$$\eta_V^l = \eta_V^r : \mathbb{R} \to V \otimes V$$

$$1 \mapsto \sum_i e_i \otimes e_i$$

$$\epsilon_V^l = \epsilon_V^r : V \otimes V \to \mathbb{R}$$

$$\sum_{ij} c_{ij}\, v_i \otimes w_i \mapsto \sum_{ij} c_{ij} \langle v_i | w_i \rangle$$

$\mathbf{FVect} \times \mathbf{P}$ – categorical representation of meaning space

Objects in $\mathbf{FVect}$ are of the form (V, p), where V is the vector space representation of the meaning and p is the pregroup type. There exists a morphism $(f, \leq) : (V, p) \to (W, q)$ if there exists a morphism $f : V \to W$ in $\mathbf{FVect}$ and $p \leq q$ in $\mathbf{P}$.

The compact closed structure of $\mathbf{FVect}$ and $\mathbf{P}$ lifts componentwise to the product category $\mathbf{FVect} \times \mathbf{P}$:

$$\eta^l : (\mathbb{R}, 1) \to (V \otimes V, p \cdot p^l)$$

$$\eta^r : (\mathbb{R}, 1) \to (V \otimes V, p^r \cdot p)$$

$$\epsilon^l : (V \otimes V, p^l \cdot p) \to (\mathbb{R}, 1)$$

$$\epsilon^r : (V \otimes V, p \cdot p^r) \to (\mathbb{R}, 1)$$

Definition 6.3. An object (V, p) in the product category is called a **meaning space**, where V is the vector space in which the meanings $\vec{v} \in V$ of strings of type p live.

“From-the-meanings-of-words-to-the-meanings-of-the-sentence” map

- Let $v_1 v_2 \dots v_n$ be a string of words, each v_i with a meaning space representation $\overrightarrow{v_i} \in (V_i, p_i)$. Let $\mathrm{x} \in P$ be a pregroup type such that $[p_1 p_2 \dots p_n \leq x]$. Then the meaning vector for the string is:

$$\overrightarrow{v_1 v_2 \dots v_n} \in (W, x) \equiv f(v_1 \otimes v_2 \otimes \cdots \otimes v_n),$$

- where f is defined to be the application of the compact closure maps obtained from the reduction $[p_1 p_2 \dots p_n \leq x]$ to the composite vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Example: “John likes Mary”

- It has the pregroup type $nn^r s n^l n$
- vector representations $\overrightarrow{John}, \overrightarrow{Mary} \in V$ and $\overrightarrow{likes} \in V \otimes S \otimes V$
- The morphism in $\mathbf{FVect} \times \mathbf{P}$ corresponding to the map is of type:

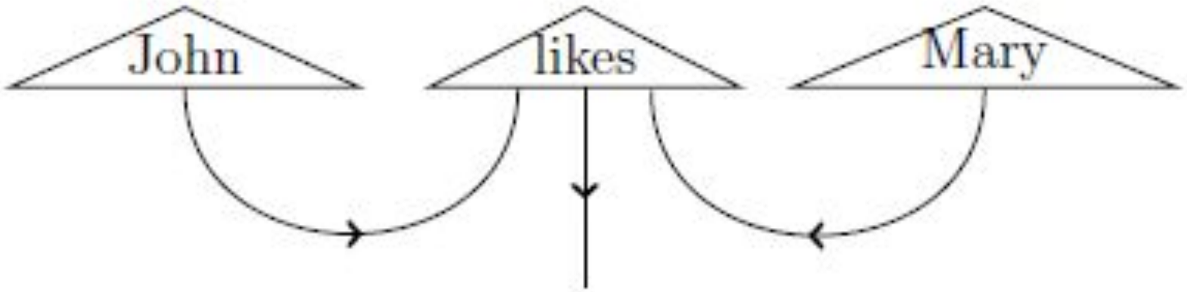
$$\left(V \otimes (V \otimes S \otimes V) \otimes V, nn^r s n^l n\right) \to (s, S)$$

- From the pregroup reduction $[nn^r s n^l n \to s]$ we obtain the compact closure maps $\epsilon^r 1 \epsilon^l$. In $\mathbf{FVect}$ this translates into:

$$\epsilon_V \otimes 1_S \otimes \epsilon_V : V \otimes (V \otimes S \otimes V) \otimes V \to S$$

Example: “John likes Mary”

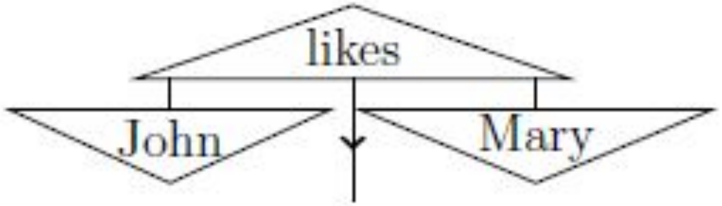
- $\overrightarrow{John} \otimes \overrightarrow{likes} \otimes \overrightarrow{Mary}$



$$\overrightarrow{likes} = \sum_{ijk} c_{ijk} v_i \otimes s_j \otimes v_k$$

$$\begin{aligned}\overrightarrow{John\ likes\ Mary} &= \epsilon_V \otimes 1_S \otimes \epsilon_V (\overrightarrow{John} \otimes \overrightarrow{likes} \otimes \overrightarrow{Mary}) \\ &= \sum_{ijk} \langle John|v_i\rangle s_j \langle v_k|Mary\rangle\end{aligned}$$

$$(\langle \epsilon_V^r| \otimes 1_S \otimes \langle \epsilon_V^l|) \circ |\overrightarrow{John} \otimes \overrightarrow{likes} \otimes \overrightarrow{Mary}\rangle$$



$$(\langle \overrightarrow{John}| \otimes 1_S \otimes \langle \overrightarrow{Mary}|) \circ |\overrightarrow{likes}\rangle$$

Example

$$
\begin{aligned}
\ulcorner eat \urcorner =& (|\overrightarrow{sloths}\rangle|\overrightarrow{plants}\rangle + |\overrightarrow{lions}\rangle|\overrightarrow{meat}\rangle)(\langle\overrightarrow{sloths}|\langle\overrightarrow{plants}| + \langle\overrightarrow{lions}|\langle\overrightarrow{meat}|) \\
=& (|\overrightarrow{sloths}\rangle|\overrightarrow{plants}\rangle)(\langle\overrightarrow{sloths}|\langle\overrightarrow{plants}|)+ \\
& \quad (|\overrightarrow{sloths}\rangle|\overrightarrow{plants}\rangle)(\langle\overrightarrow{lions}|\langle\overrightarrow{meat}|)+ \\
& \quad (|\overrightarrow{lions}\rangle|\overrightarrow{meat}\rangle)(\langle\overrightarrow{sloths}|\langle\overrightarrow{plants}|)+ \\
& \quad (|\overrightarrow{lions}\rangle|\overrightarrow{meat}\rangle)(\langle\overrightarrow{lions}|\langle\overrightarrow{meat}|) \\
\sim& (|\overrightarrow{sloths}\rangle\langle\overrightarrow{sloths}| \otimes |\overrightarrow{plants}\rangle\langle\overrightarrow{plants}|)+ \\
& \quad (|\overrightarrow{sloths}\rangle\langle\overrightarrow{lions}| \otimes |\overrightarrow{plants}\rangle\langle\overrightarrow{meat}|)+ \\
& \quad (|\overrightarrow{lions}\rangle\langle\overrightarrow{sloths}| \otimes |\overrightarrow{meat}\rangle\langle\overrightarrow{plants}|)+ \\
& \quad (|\overrightarrow{lions}\rangle\langle\overrightarrow{lions}| \otimes |\overrightarrow{meat}\rangle\langle\overrightarrow{meat}|)
\end{aligned}
$$

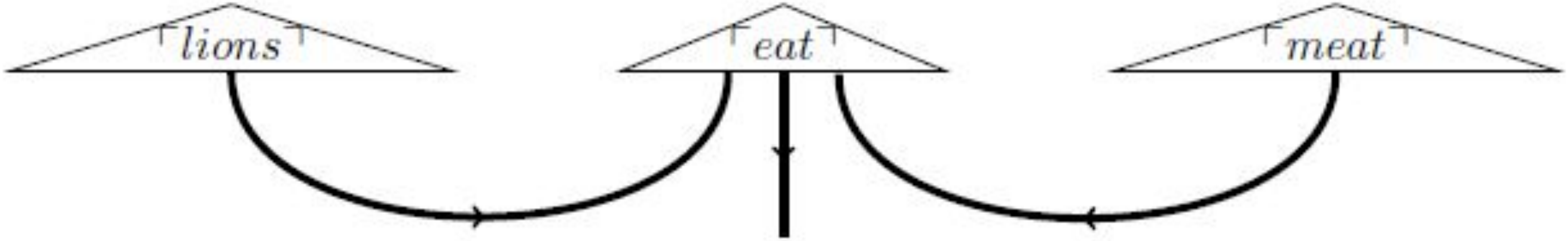
Example

This is the density matrix representation of a pure composite state that relate "sloths" to "plants" and "lions" to "meat". If we fix the bases $\{\overrightarrow{lions}, \overrightarrow{sloths}\}$ for N_1, and $\{\overrightarrow{meat}, \overrightarrow{plants}\}$ for N_2, $\ulcorner eat \urcorner : N_1 \otimes N_1 \to N_2 \otimes N_2$ has the following matrix representation:

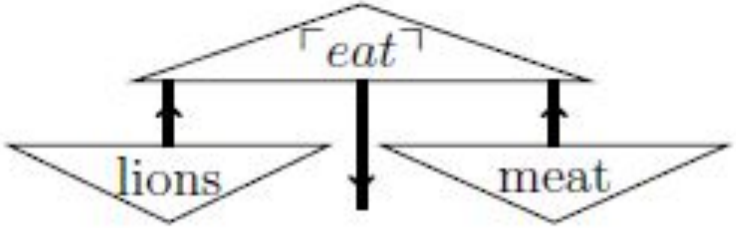
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

"Lions eat meat" . This is a transitive sentence, so as before, it gets assigned the pregroup type: $nn^l s n^r n$. The diagrammatic expression of the pregroup reduction is as follows:

Example



This reduces to:



Example

Explicit calculation gives:

$$
\begin{aligned}
(\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)(\ulcorner lions \urcorner \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) \\
&= \langle \overrightarrow{lions} | \overrightarrow{sloths} \rangle^2 \langle \overrightarrow{plants} | \overrightarrow{meat} \rangle^2 + \\
&\quad \langle \overrightarrow{lions} | \overrightarrow{sloths} \rangle \langle \overrightarrow{lions} | \overrightarrow{lions} \rangle \langle \overrightarrow{meat} | \overrightarrow{meat} \rangle \langle \overrightarrow{plants} | \overrightarrow{meat} \rangle + \\
&\quad \langle \overrightarrow{lions} | \overrightarrow{lions} \rangle \langle \overrightarrow{lions} | \overrightarrow{sloths} \rangle \langle \overrightarrow{meat} | \overrightarrow{meat} \rangle \langle \overrightarrow{plants} | \overrightarrow{meat} \rangle + \\
&\quad \langle \overrightarrow{lions} | \overrightarrow{lions} \rangle^2 \langle \overrightarrow{meat} | \overrightarrow{meat} \rangle^2 \\
&= 0 + 0 + 0 + 1 \\
&= 1
\end{aligned}
$$

Example

"Sloths eat meat" . This sentence has a very similar calculation to the one above, and has the result:

$$(\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)(\ulcorner sloths \urcorner \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) = 0$$

"Mammals eat meat" . This sentence has the same pregroup types as the first sentence, and so has the same reduction map:

$$\begin{aligned}
&(\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)(\ulcorner mammals \urcorner \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) \\
&\qquad = (\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)((\frac{1}{2}\ulcorner lions \urcorner + \frac{1}{2}\ulcorner sloths \urcorner) \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) \\
&\qquad = \frac{1}{2}(\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)(\ulcorner lions \urcorner \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) + \\
&\qquad\quad \frac{1}{2}(\epsilon_N^l \otimes 1_S \otimes \epsilon_N^r)(\ulcorner sloths \urcorner \otimes \ulcorner eat \urcorner \otimes \ulcorner meat \urcorner) \\
&\qquad = \frac{1}{2}
\end{aligned}$$

Readings

- Esma Balkir. Using Density Matrices in a Compositional Distributional Model of Meaning. // Master thesis. University of Oxford. 2014
- Joachim Lambek. Type grammars as pregroups. Grammars, 4(1):21{39, 2001.