Made by ALEX

"Pythagoras' theorem and similar shapes"

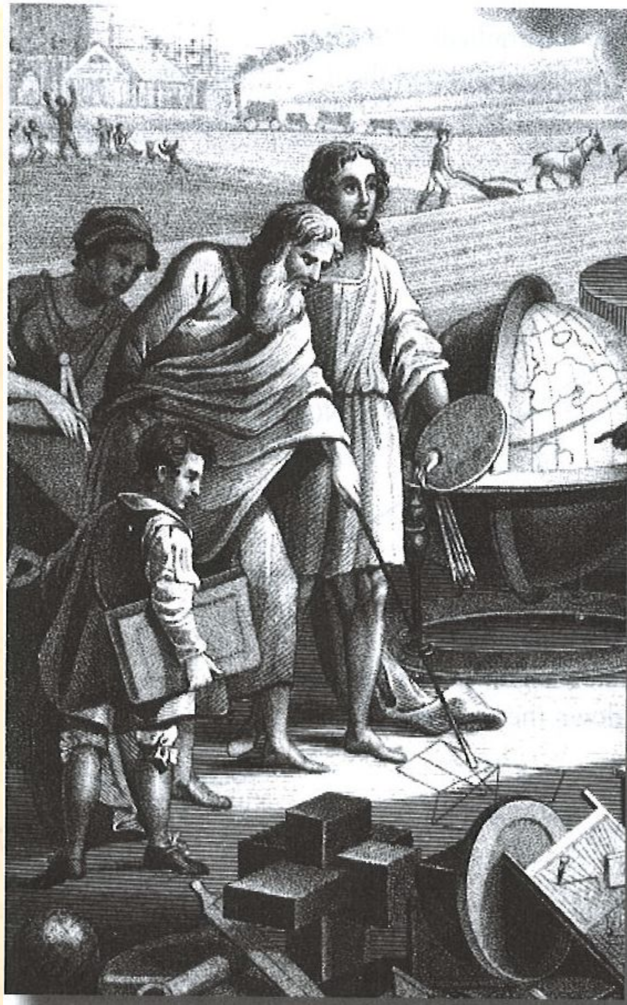
11 Pythagoras' theorem and similar shapes

Key words

- Right angle
- Hypotenuse
- Similar
- Corresponding sides
- Corresponding angles
- Scale factor of lengths
- Scale factor of volumes
- Scale factor of areas
- Congruent
- Included side
- Included angle

In this chapter you will learn how to:

- use Pythagoras' theorem to find unknown sides of a right-angled triangles
- learn how to use Pythagoras' theorem to solve problems
- decide whether or not triangles are mathematically similar
- use properties of similar triangles to solve problems



Pythagoras' theorem

Learning the rules

Pythagoras' theorem describes the relationship between the sides of a right-angled triangle.

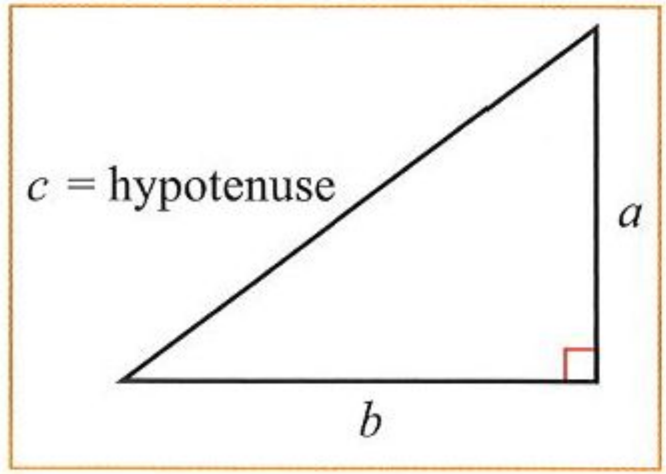
The longest side – the side that does not touch the right angle – is known as the **hypotenuse**.

For this triangle, Pythagoras' theorem states that: $\boxed{a^2 + b^2 = c^2}$

In words this means that the square on the hypotenuse is equal to the sum of the squares on the other two sides. Notice that the square of the hypotenuse is the subject of the equation. This should help you to remember where to place each number.

Tip

You will be expected to remember Pythagoras' theorem.



Worked example 1

Find the value of x in each of the following triangles, giving your answer to one decimal place.

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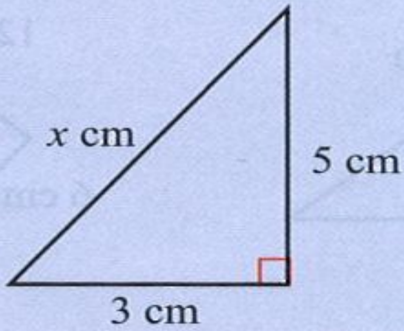

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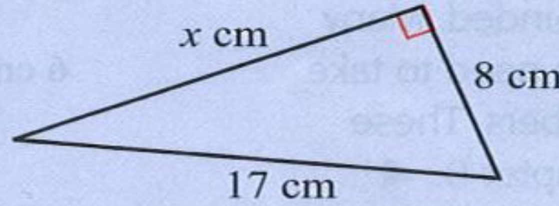


(a)

$$a^2 + b^2 = c^2$$
$$3^2 + 5^2 = x^2$$
$$9 + 25 = x^2$$
$$\Rightarrow x^2 = 34$$
$$x = \sqrt{34} = 5.8309\ldots$$
$$\approx 5.8\,\text{cm (1dp)}$$

Notice that the final answer needs to be rounded.

(b)

$$a^2 + b^2 = c^2$$
$$8^2 + x^2 = 17^2$$
$$64 + x^2 = 289$$
$$x^2 = 289 - 64$$
$$x^2 = 225$$
$$x = \sqrt{225} = 15\,\text{cm (1dp)}$$

Notice that a shorter side needs to be found so, after writing the Pythagoras formula in the usual way, the formula has to be rearranged to make x^2 the subject.

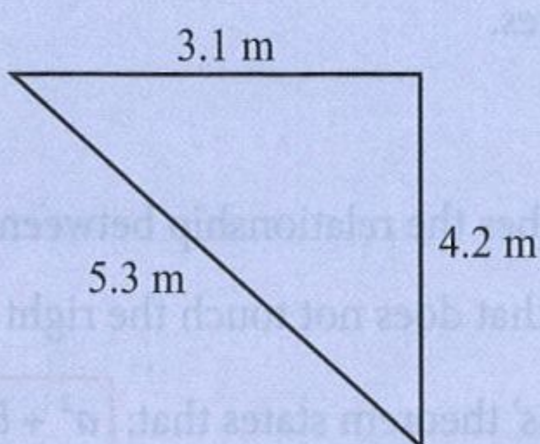
Checking for right-angled triangles

You can also use the theorem to determine if a triangle is right-angled or not. Substitute the values of a, b and c of the triangle into the formula and check to see if it fits. If $a^2 + b^2$ does not equal c^2 then it is *not* a right-angled triangle.

Notice here the theorem is written as $c^2 = a^2 + b^2$; you will see it written like this or like $a^2 + b^2 = c^2$ in different places but it means the same thing.

Worked example 2

Use Pythagoras' theorem to decide whether or not the triangle shown below is right-angled.



Check to see if Pythagoras' theorem is satisfied:

$c^2 = a^2 + b^2$

$3.1^2 + 4.2^2 = 27.25$

$5.3^2 = 28.09 \neq 27.25$

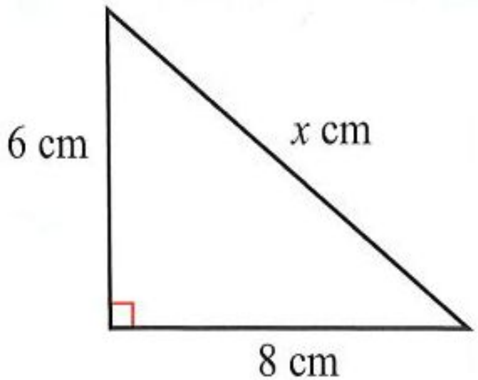
The symbol '≠' means 'does not equal'.

Pythagoras' theorem is not satisfied, so the triangle is not right-angled.

Exercise 11.1

1 Find the length of the hypotenuse in each of the following triangles.

(a)



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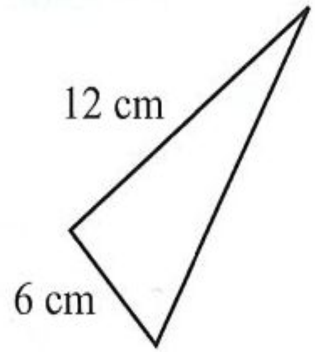

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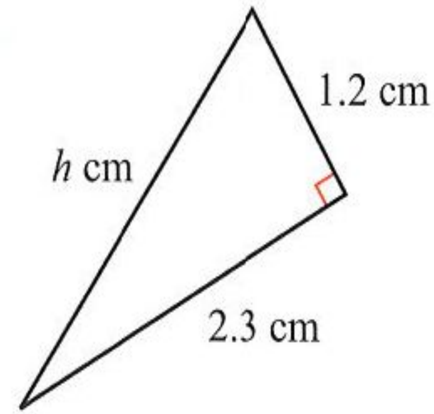

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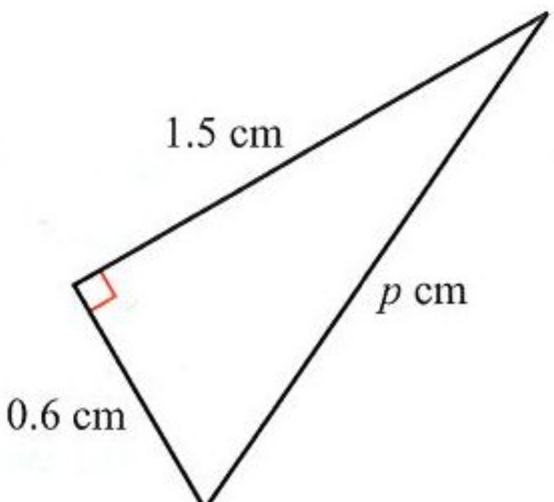

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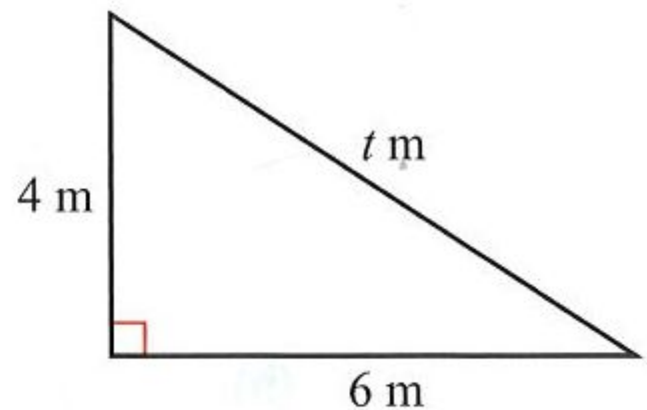
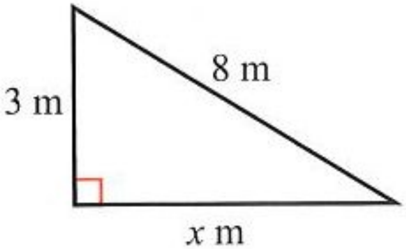


2 Find the values of the unknown lengths in each of the following triangles.

(a)



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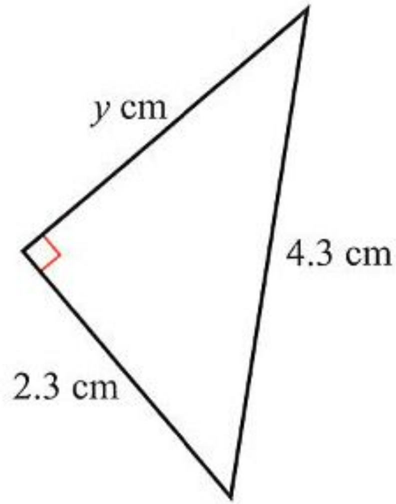

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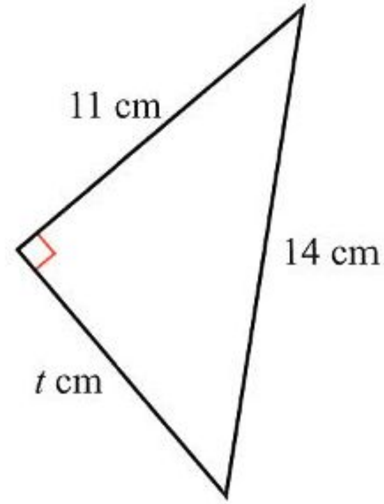

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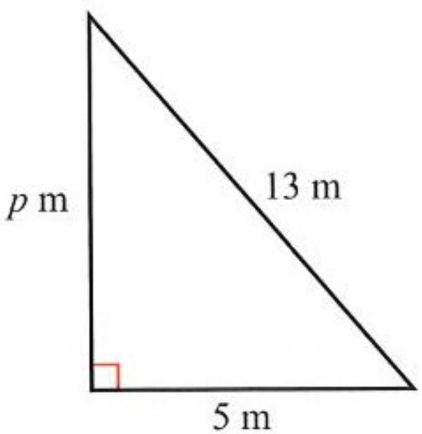

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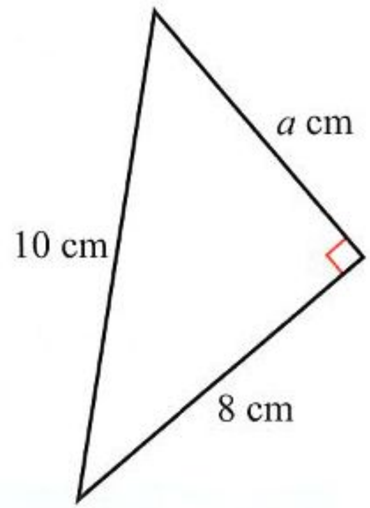
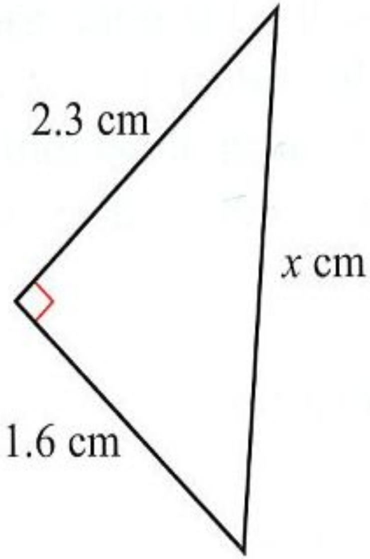


3 Find the values of the unknown lengths in each of the following triangles.

(a)



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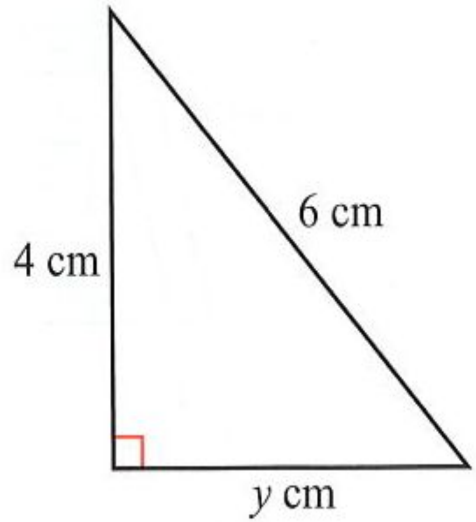

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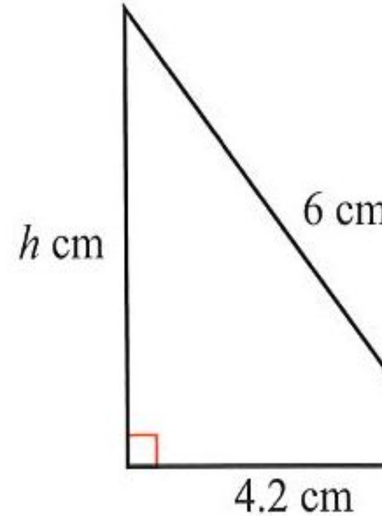

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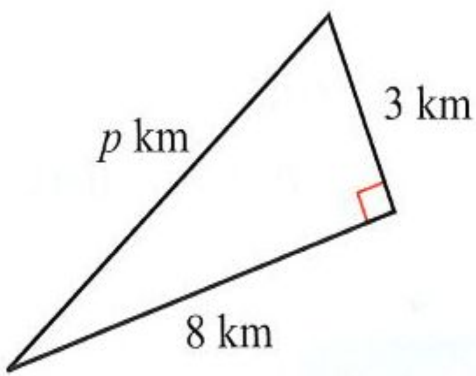

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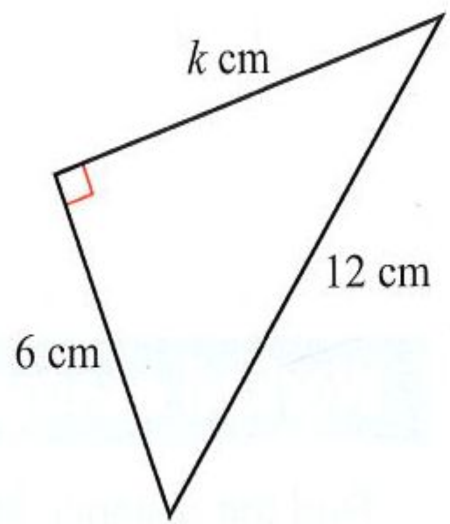

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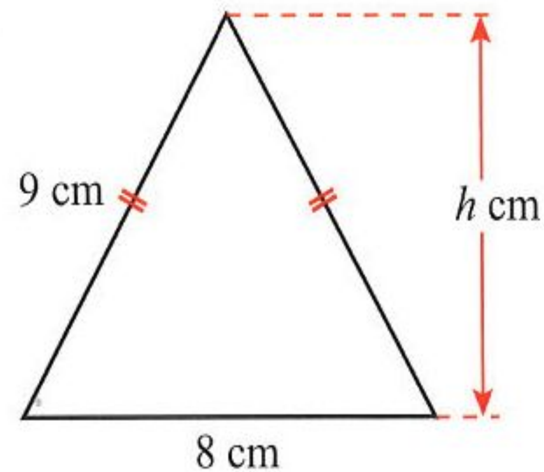
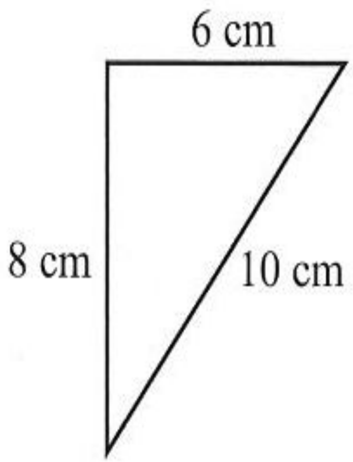


4 Use Pythagoras' theorem to help you decide which of the following triangles are right-angled.

(a)



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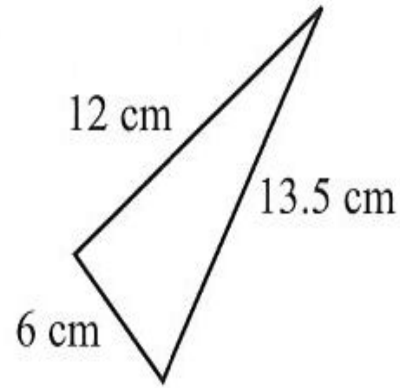

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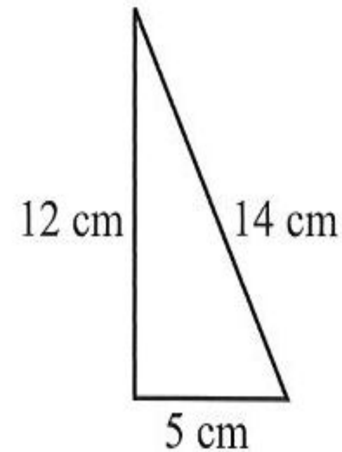

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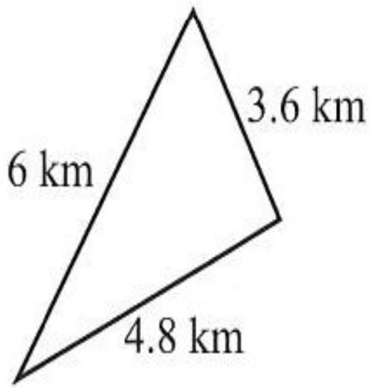

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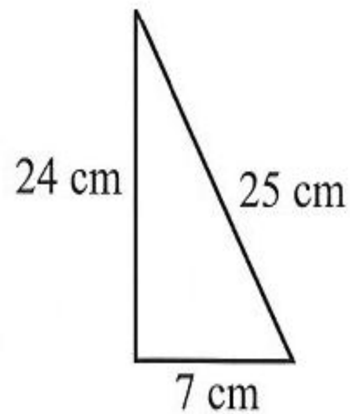
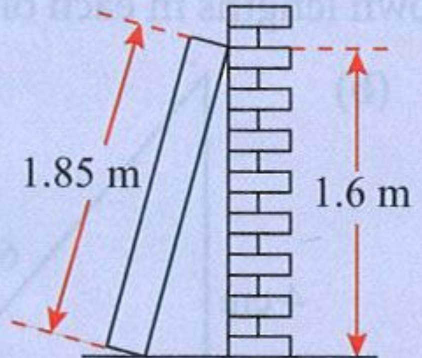
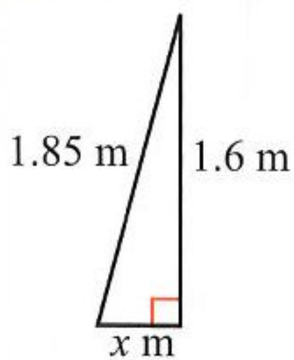


Applications of Pythagoras' theorem

Worked example 3



The diagram shows a bookcase that has fallen against a wall. If the bookcase is 1.85 m tall, and it now touches the wall at a point 1.6 m above the ground, calculate the distance of the foot of the bookcase from the wall. Give your answer to 2 decimal places.



Apply Pythagoras' theorem:

$$a^2 + b^2 = c^2$$

$$x^2 + 1.6^2 = 1.85^2$$

$$x^2 = 1.85^2 - 1.6^2$$

$$= 3.4225 - 2.56$$

$$= 0.8625$$

$$x = \sqrt{0.8625} = 0.93\,\text{m (2dp)}$$

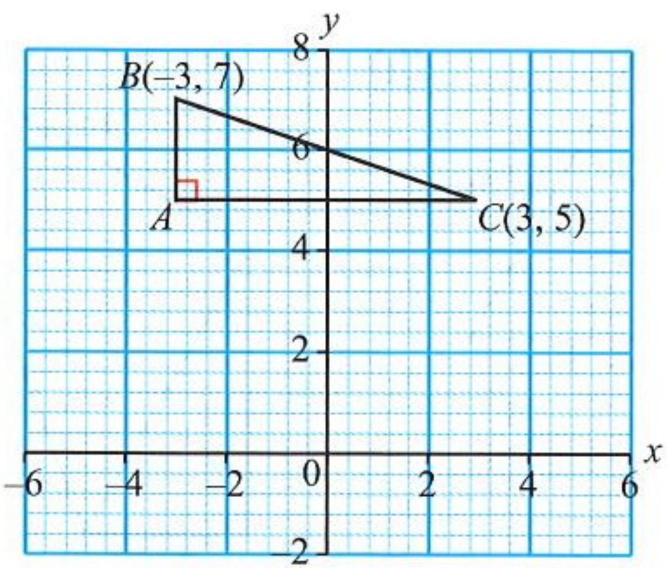
Think what triangle the situation would make and then draw it. Label each side and substitute the correct sides into the formula.

Applications of Pythagoras' theorem

It can be helpful to draw diagrams when you are given co-ordinates.

Worked example 4

Find the distance between the points $A(3, 5)$ and $B(-3, 7)$.



$AB = 7 - 5 = 2$ units — Difference between y-co-ordinates.

$AC = 3 - -3 = 6$ units — Difference between x-co-ordinates.

$$\begin{aligned} BC^2 &= 2^2 + 6^2 \\ &= 4 + 36 \\ &= 40 \end{aligned}$$

Apply Pythagoras' theorem.

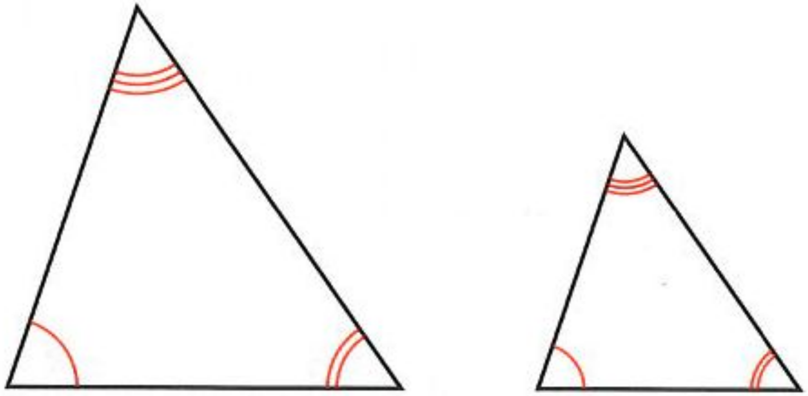
$$\begin{aligned} \text{So } BC &= \sqrt{40} \\ &= 6.32 \text{ units (3sf)} \end{aligned}$$

Understanding similar triangles

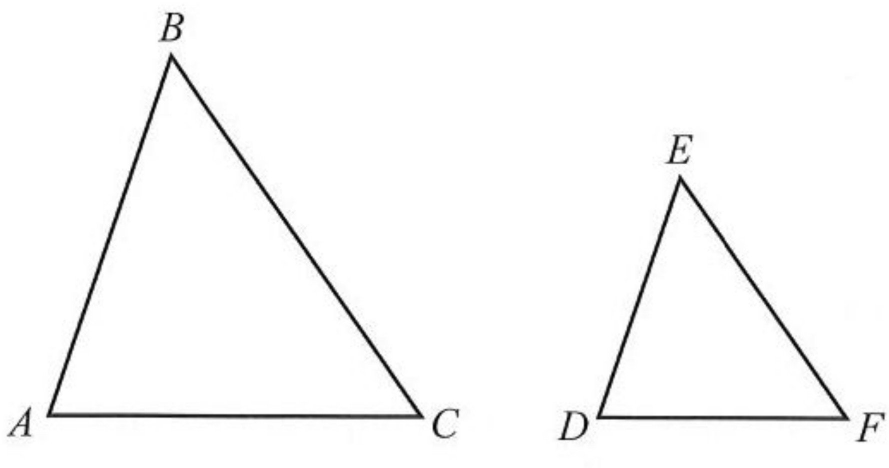
Two mathematically **similar** objects have exactly the same shape and proportions, but may be different in size.

When one of the shapes is enlarged to produce the second shape, each part of the original will *correspond* to a particular part of the new shape. For triangles, **corresponding sides** join the same angles.

All of the following are true for similar triangles:

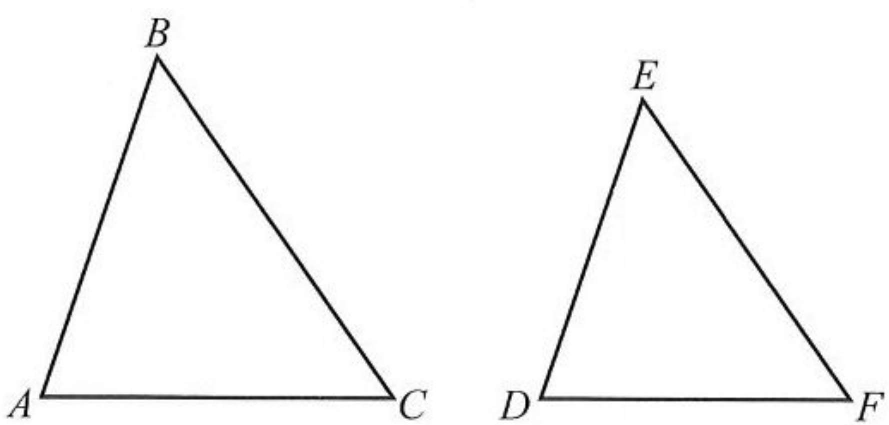


Corresponding angles are equal.



'Internal' ratios of sides are the same for both triangles. For example:

$$\frac{AB}{BC}=\frac{DE}{EF}$$

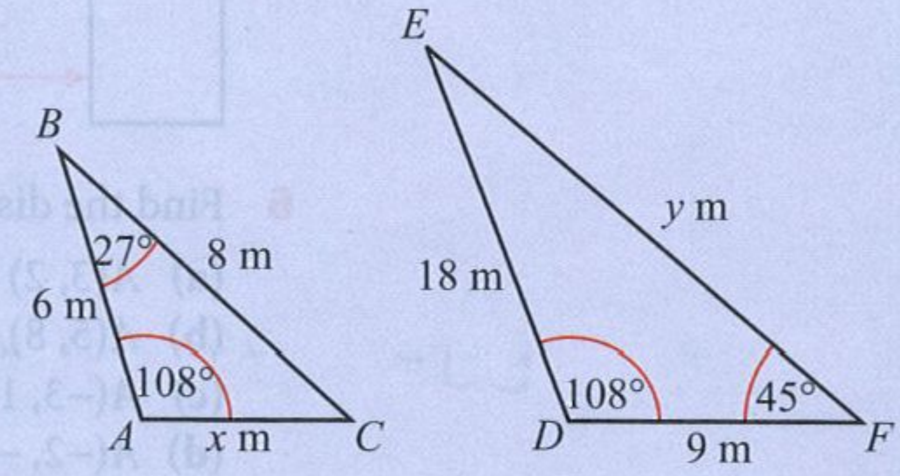


Ratios of corresponding sides are equal:

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

Worked example 5

Explain why the two triangles shown in the diagram are similar and work out x and y.



$A\hat{C}B = 180° - 27° - 108° = 45°$

$F\hat{E}D = 180° - 45° - 108° = 27°$

So both triangles have exactly the same three angles and are, therefore, similar.

Since the triangles are similar: $\dfrac{DE}{AB} = \dfrac{EF}{BC} = \dfrac{DF}{AC}$

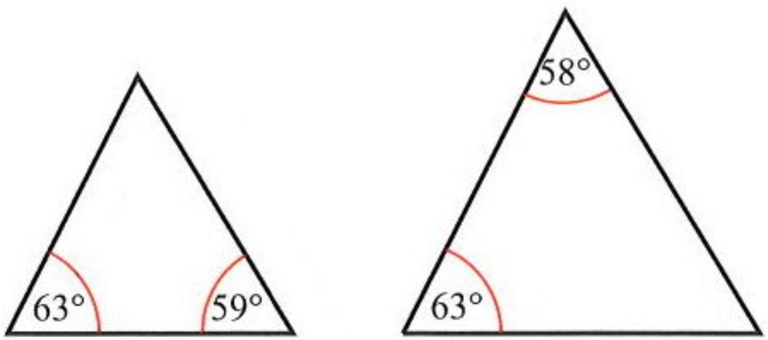
So : $\dfrac{y}{8} = \dfrac{18}{6} = 3 \Rightarrow y = 24\,\text{m}$

and: $\dfrac{9}{x} = \dfrac{18}{6} = 3 \Rightarrow x = 3\,\text{m}$

Exercise 11.3

1 For each of the following decide whether or not the triangles are similar in shape. Each decision should be explained fully.

(a)



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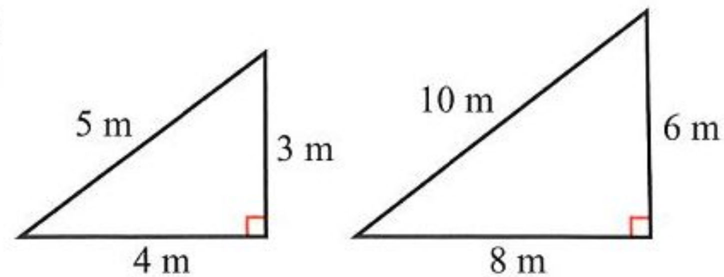

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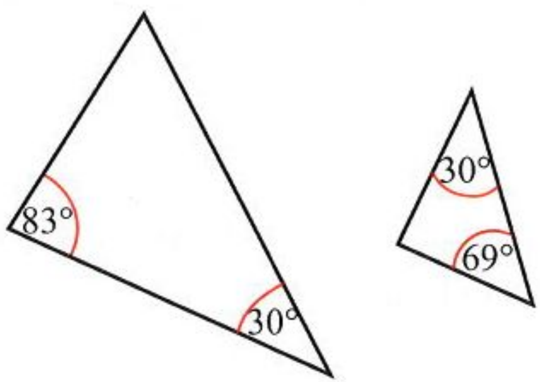

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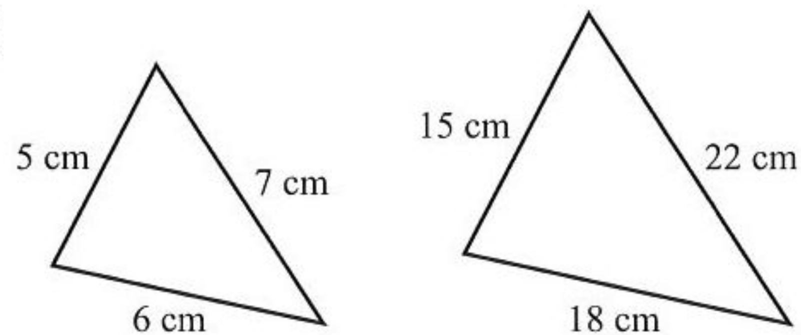

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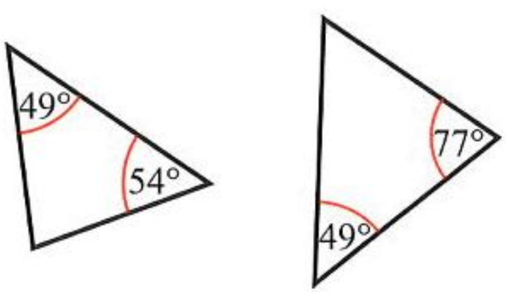

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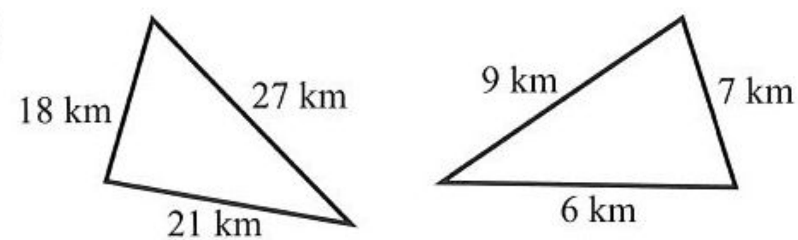
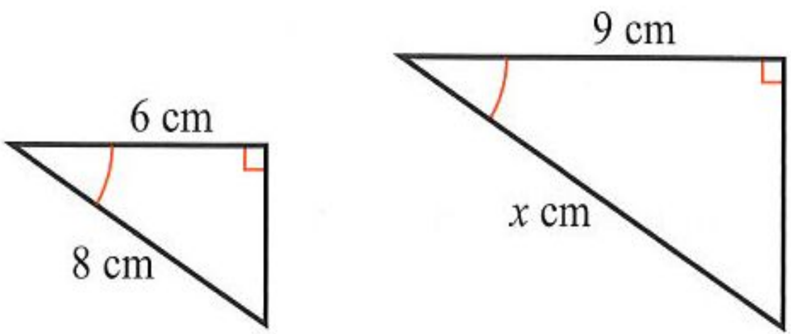


2 The pairs of triangles in this question are similar. Calculate the unknown (lettered) length in each case.

(a)



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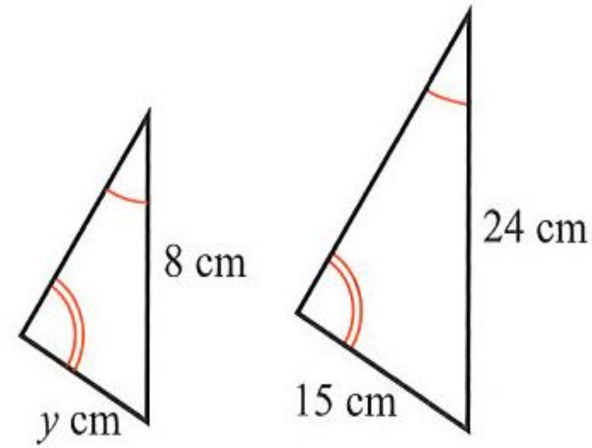

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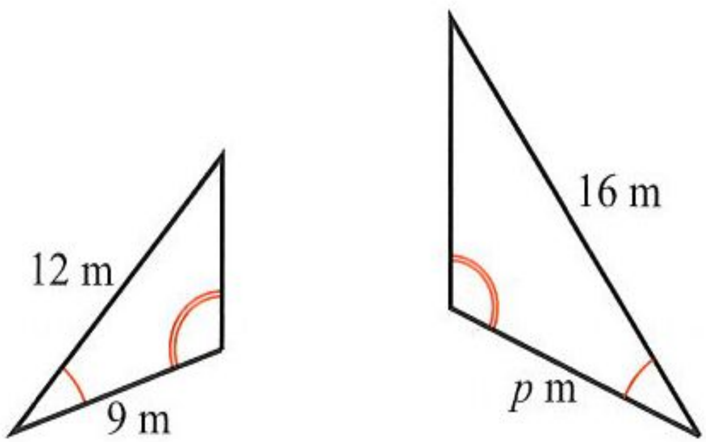

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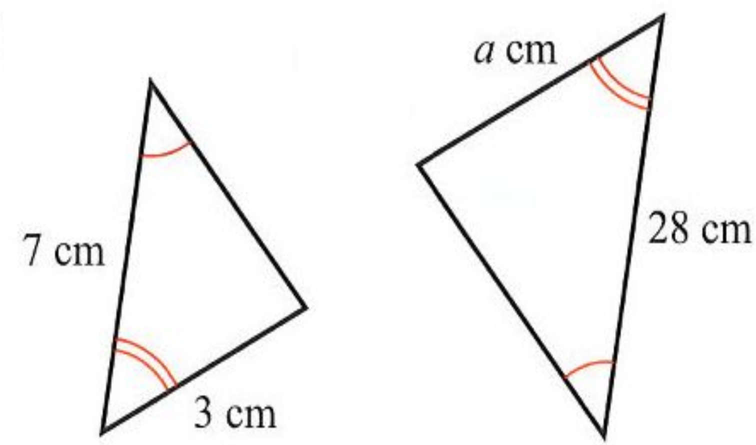
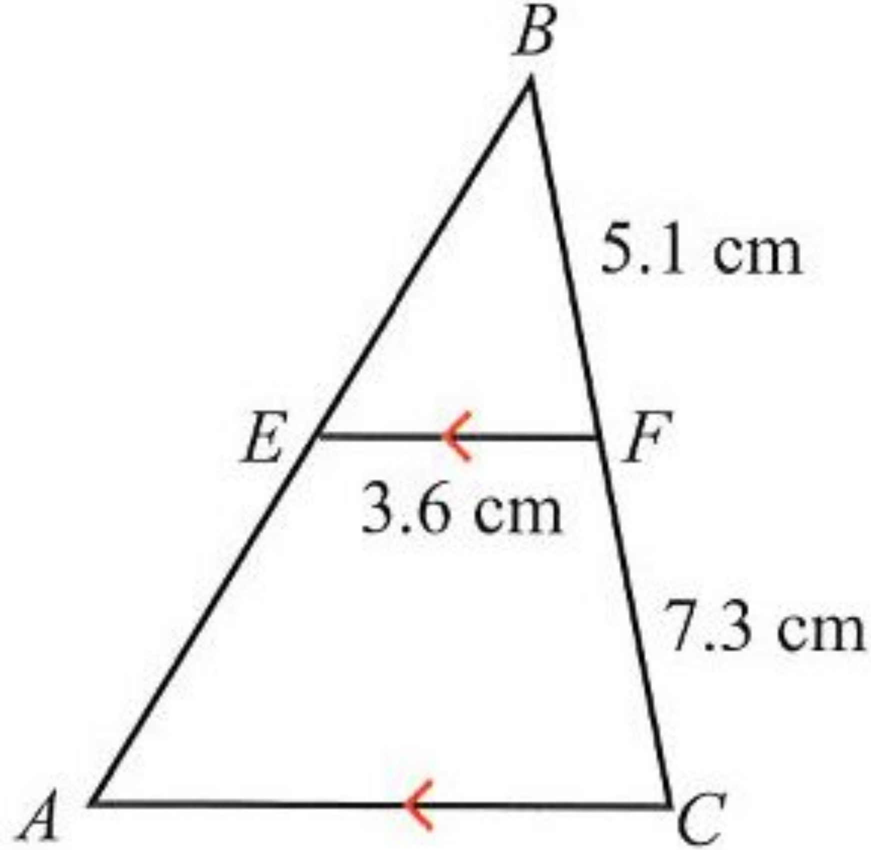
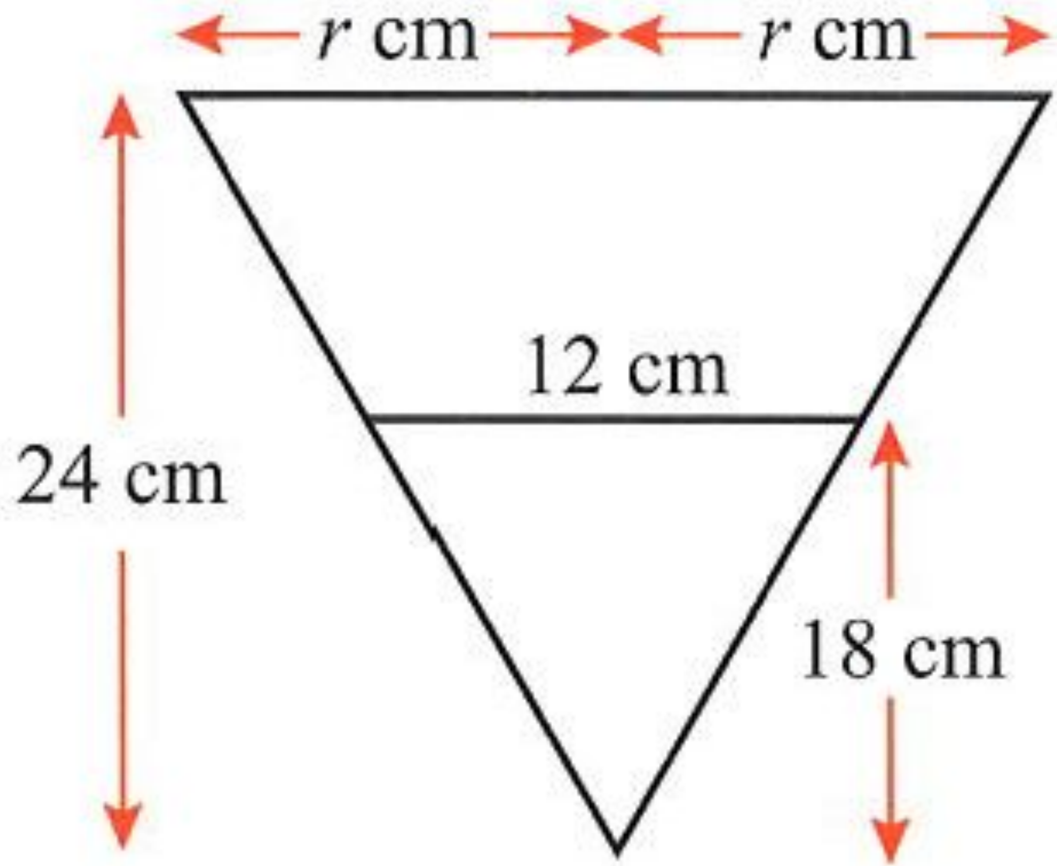


3 The diagram shows triangle ABC. If AC is parallel to EF, find the length of AC.



7 The diagram shows a circular cone that has been filled to a depth of 18 cm. Find the radius r of the top of the cone.



Understanding similar shapes

Worked example 7

Ahmed has two rectangular flags. One measures 1000 mm by 500 mm, the other measures 500 mm by 350 mm. Are the flags similar in shape?

$\frac{1000}{500} = 2$ and $\frac{500}{350} = 1.43$ (Work out the ratio of corresponding sides.)

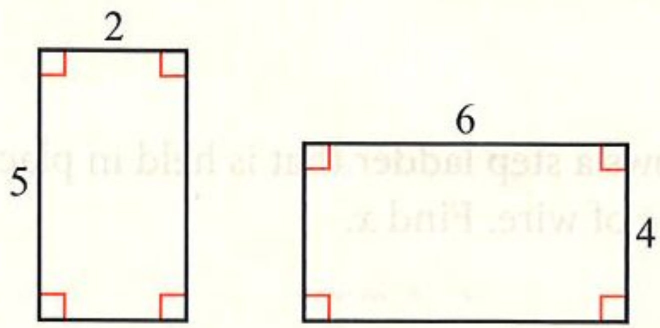
$\frac{1000}{500} \neq \frac{500}{350}$

The ratio of corresponding sides is not equal, therefore the shapes are not similar.

Exercise 11.4

1 Establish whether each pair of shapes is similar or not. Show your working.

(a)



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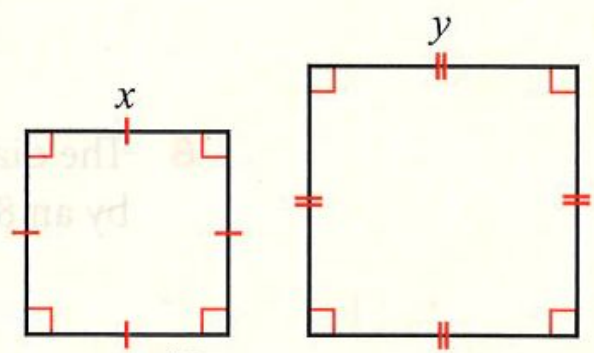

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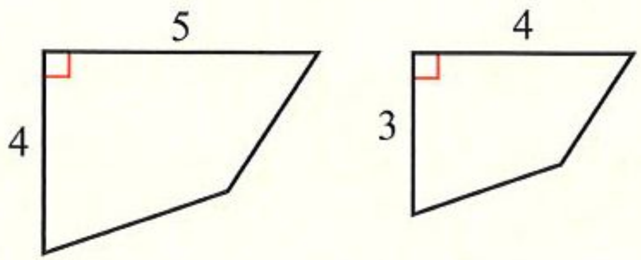

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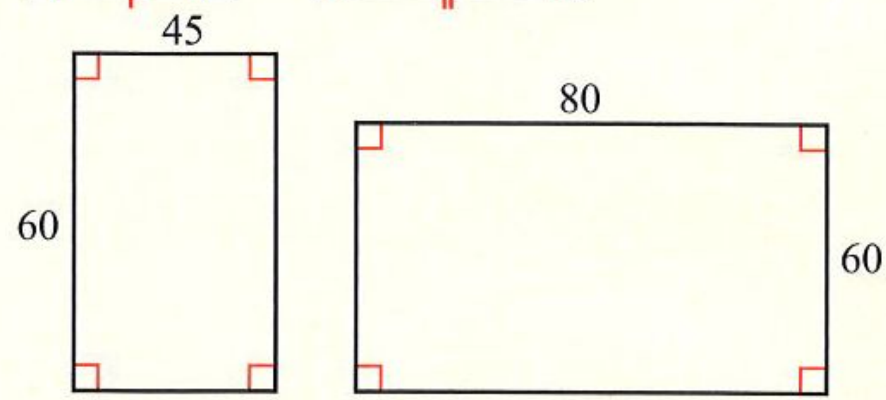

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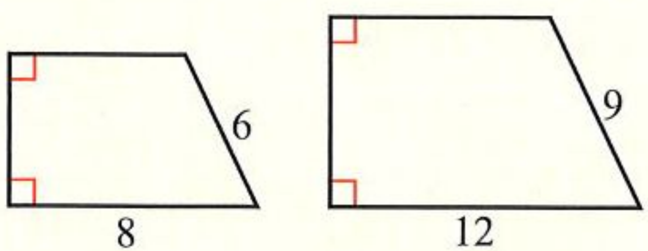

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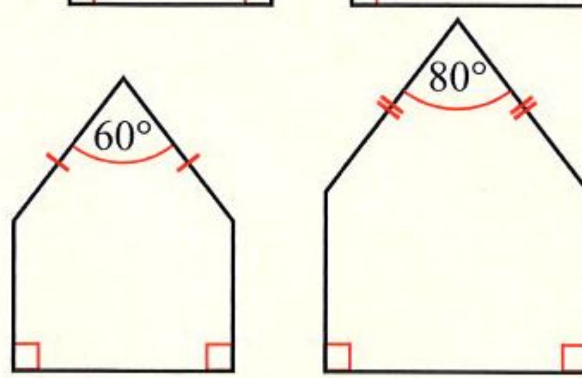
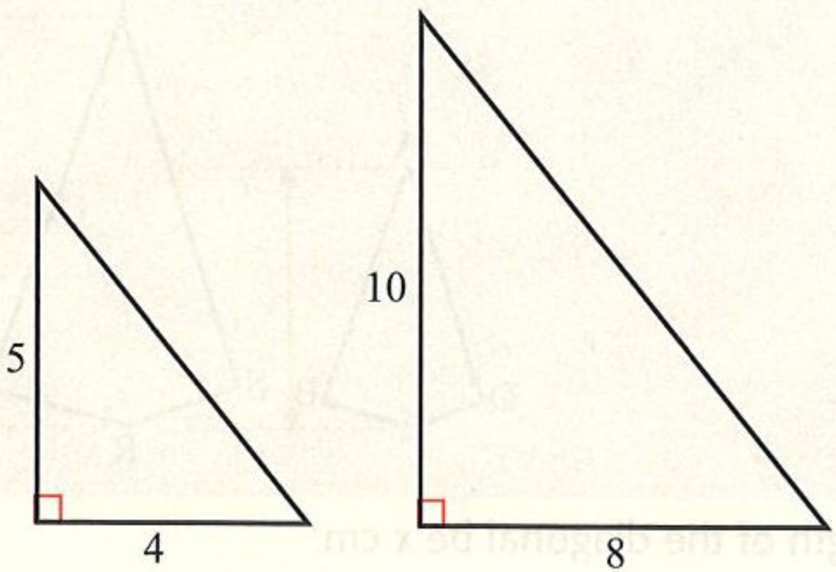


Area of similar shapes

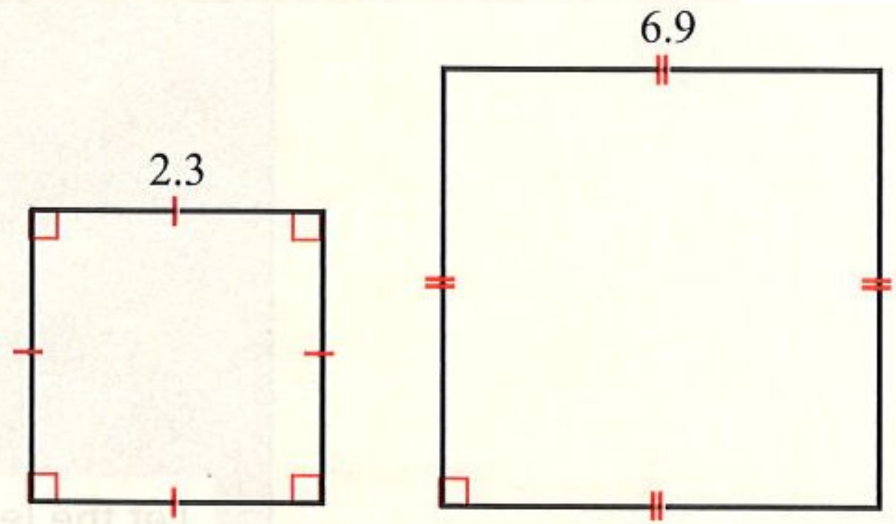
Each pair of shapes below is similar:



Area = 10   Area = 40

$$\text{Scale factor} = \frac{10}{5} = 2$$

$$\text{Area factor} = \frac{40}{10} = 4$$



Area = 5.29   Area = 47.61

$$\text{Scale factor} = \frac{6.9}{2.3} = 3$$

$$\text{Area factor} = \frac{47.61}{5.29} = 9$$

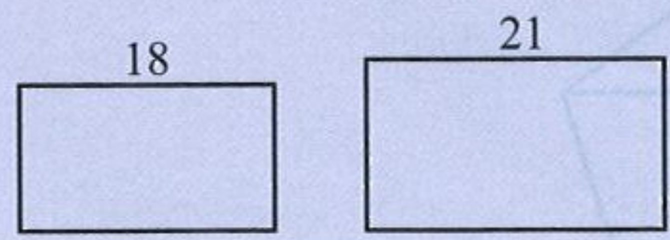
If you look at the diagrams and the dimensions you can see that there is a relationship between the corresponding sides of similar figures and the areas of the figures.

In similar figures where the ratio of corresponding sides is $a : b$, the ratio of areas is $a^2 : b^2$.

scale factor of areas = (scale factor of lengths)^2

Worked example 9

These two rectangles are similar. What is the ratio of the smaller area to the larger?



Ratio of sides = 18 : 21

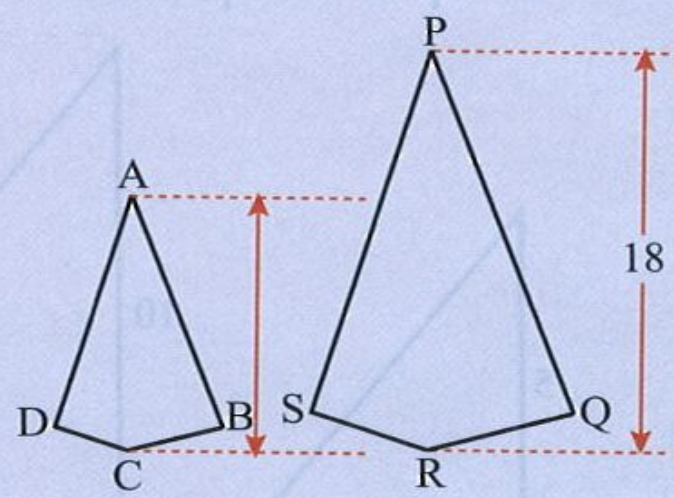
Ratio of areas = $(18)^2 : (21)^2$

= 324 : 441

= 36 : 49

Worked example 11

The shapes below are similar. Given that the area of $ABCD = 48\text{ cm}^2$ and the area of $PQRS = 108\text{ cm}^2$, find the diagonal AC in $ABCD$.



Let the length of the diagonal be x cm.

$$\frac{48}{108} = \frac{x^2}{18^2}$$

$$\frac{48}{108} = \frac{x^2}{324}$$

$$\frac{48}{108} \times 324 = x^2$$

$x^2 = 144$

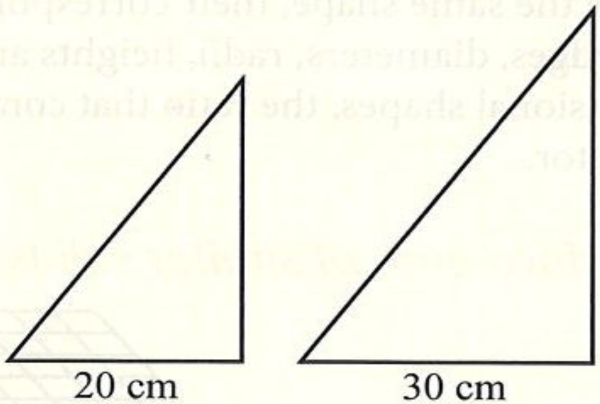
$x = 12$

Diagonal AC is 12 cm long.

Exercise 11.5

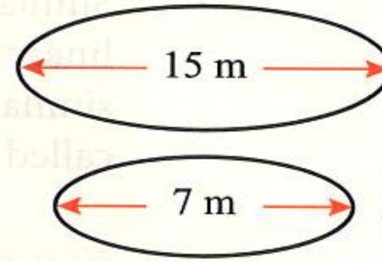
1 In each part of this question, the two figures are similar. The area of one figure is given. Find the area of the other.

(a)



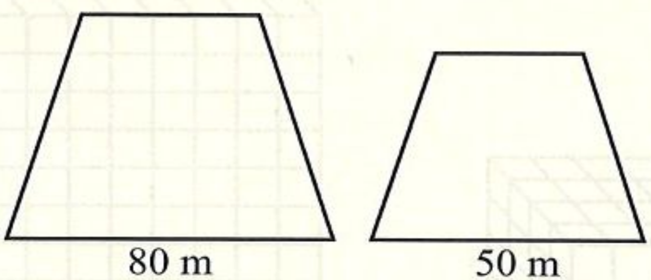
Area $= 187.5\ \text{cm}^2$

(b)



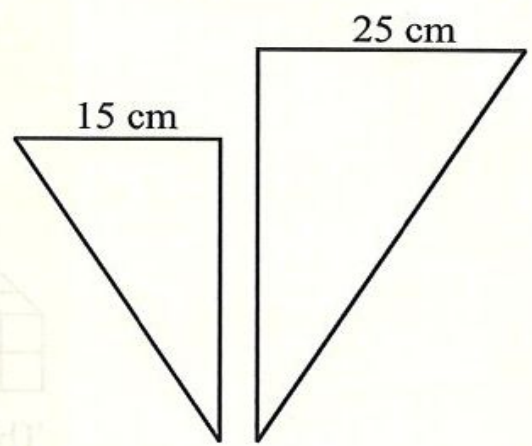
Area $= 17.0\ \text{m}^2$

(c)



Area $= 4000\ \text{m}^2$

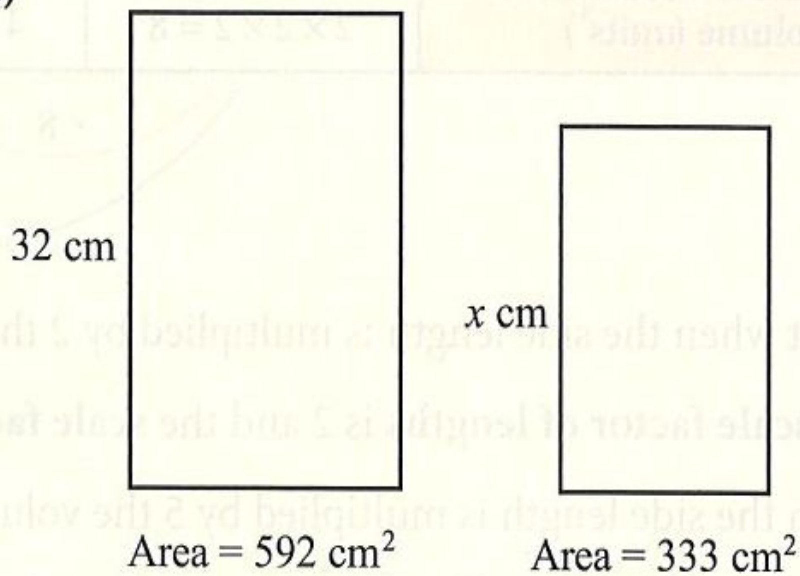
(d)



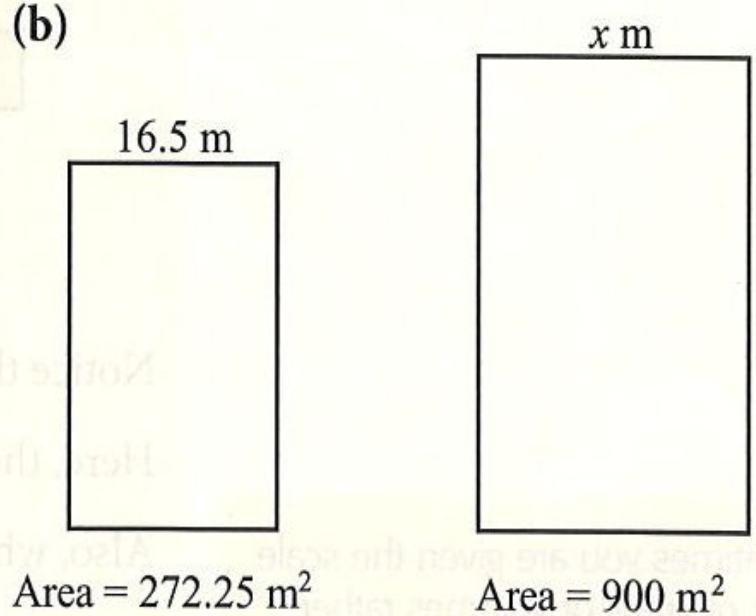
Area $= 135\ \text{cm}^2$

2 In each part of this question the areas of the two similar figures are given. Find the length of the side marked x in each.

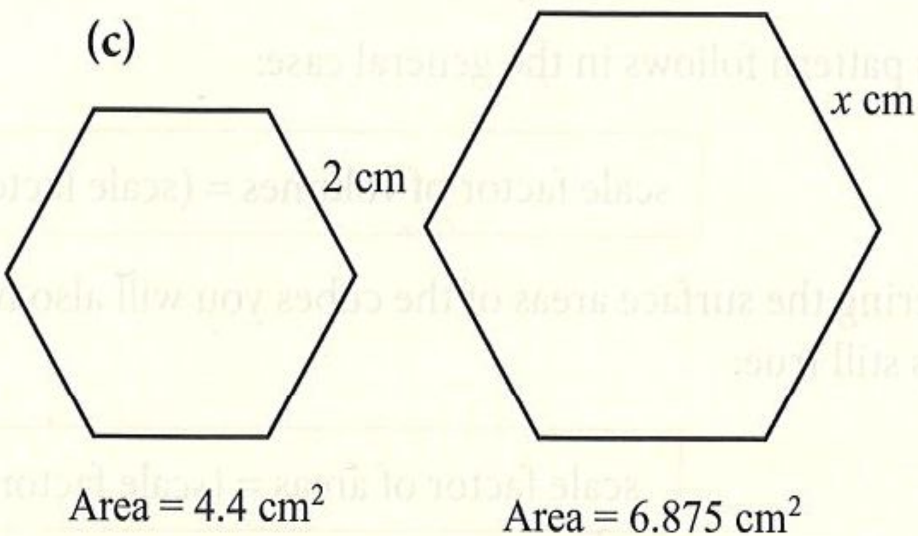
(a)



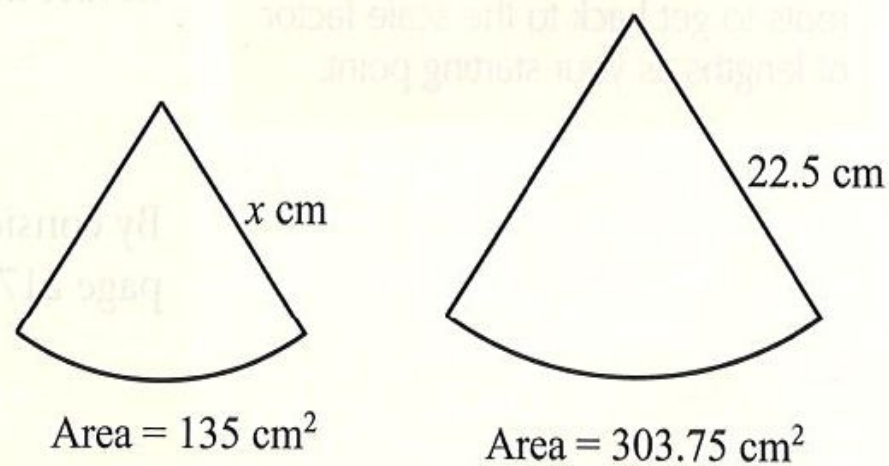
(b)



(c)

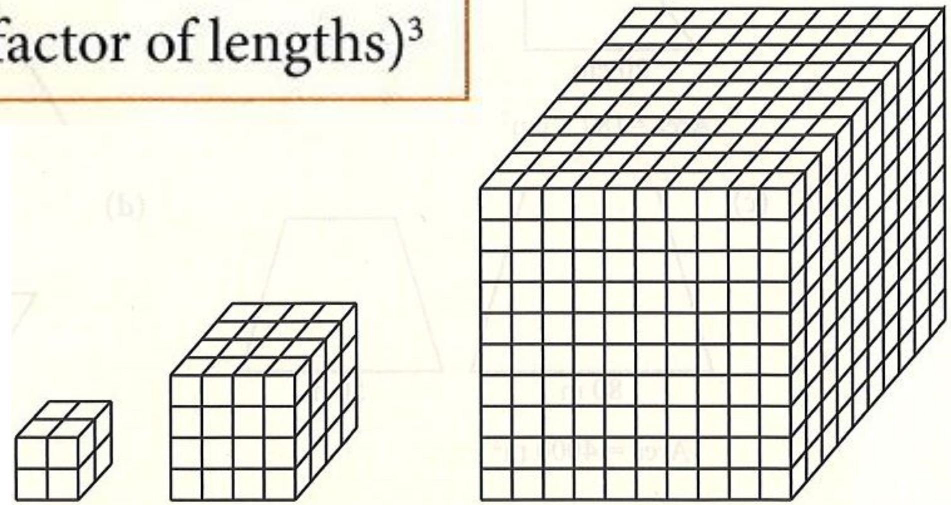


(d)



Volume and surface area of similar solids

scale factor of volumes = (scale factor of lengths)^3



$\times 5$

$\times 2$

Length of side (units)	2	4	10
Volume (units^3)	$2\times2\times2=8$	$4\times4\times4=64$	$10\times10\times10=1000$

$\times 8$

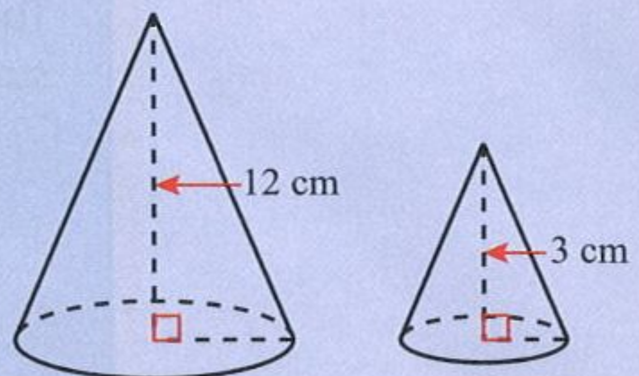
$\times 125$

In summary, if two solids (A and B) are similar:

- the ratio of their volumes is equal to the cube of the ratio of corresponding linear measures (edges, diameter, radii, heights and slant heights). In other words: Volume $A \div$ Volume $B = \left(\frac{a}{b}\right)^3$
- the ratio of their surface areas is equal to the square of the ratio of corresponding linear measures. In other words: Surface area $A \div$ Surface area $B = \left(\frac{a}{b}\right)^2$

Worked example 12

The cones shown in the diagram are mathematically similar. If the smaller cone has a volume of 40 cm³ find the volume of the larger cone.



Scale factor of lengths $= \frac{12}{3} = 4$

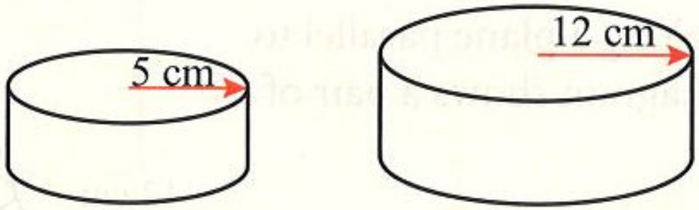
$\Rightarrow$ Scale factor of volumes $= 4^3 = 64$

So the volume of the larger cone $= 64 \times 40 = 2560\,\text{cm}^3$

Exercise 11.6

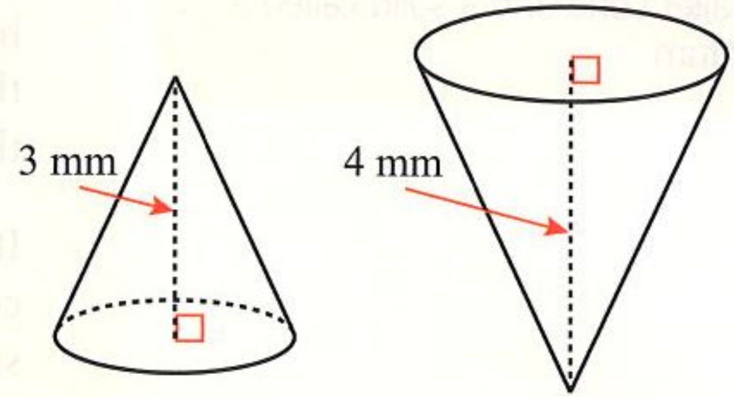
6 For each part of this question, the solids are similar. Find the unknown volume.

(a)



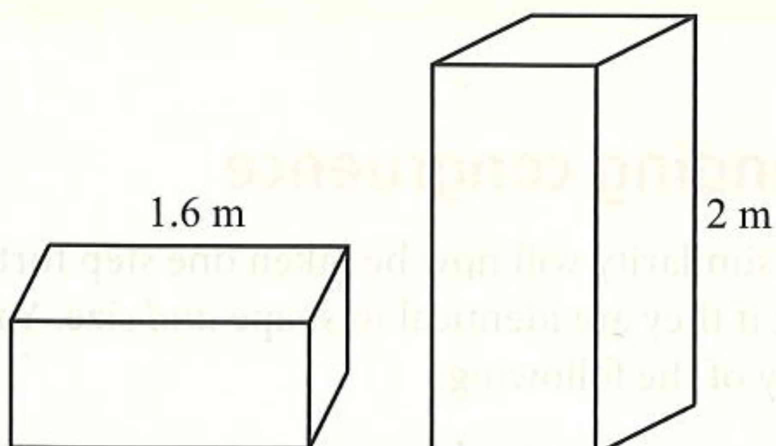
Volume = 288 cm^3

(b)



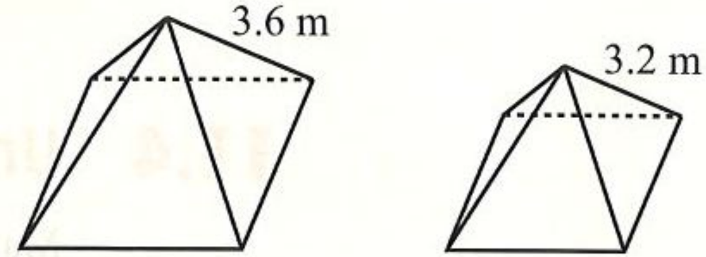
Volume = 9 mm^3

(c)



Volume = 0.384 m^3

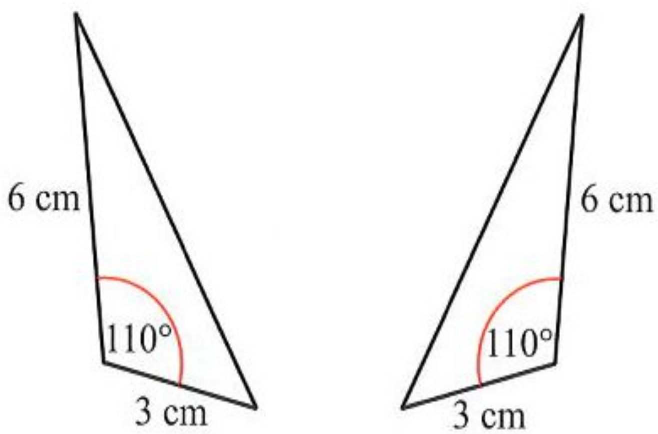
(d)



Volume = 80.64 m^3

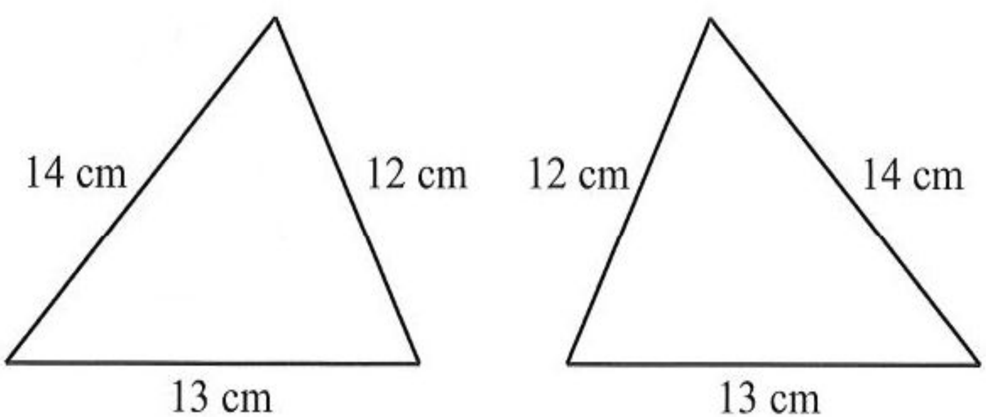
Understanding congruence

Mathematical similarity will now be taken one step further to look at congruence. Two shapes are **congruent** if they are identical in shape *and* size. You can test for congruent triangles by looking for any of the following:



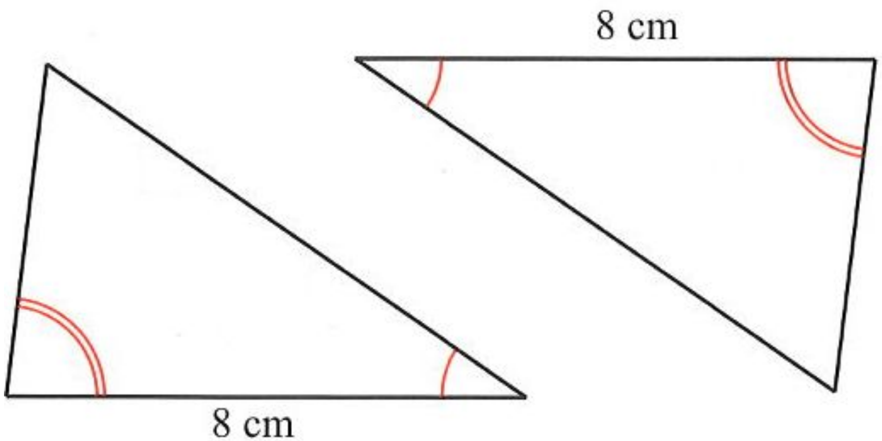
Two sides and the **included angle** (this is the angle that sits between the two given sides) are equal.

This is remembered as SAS – ***S***ide ***A***ngle ***S***ide.



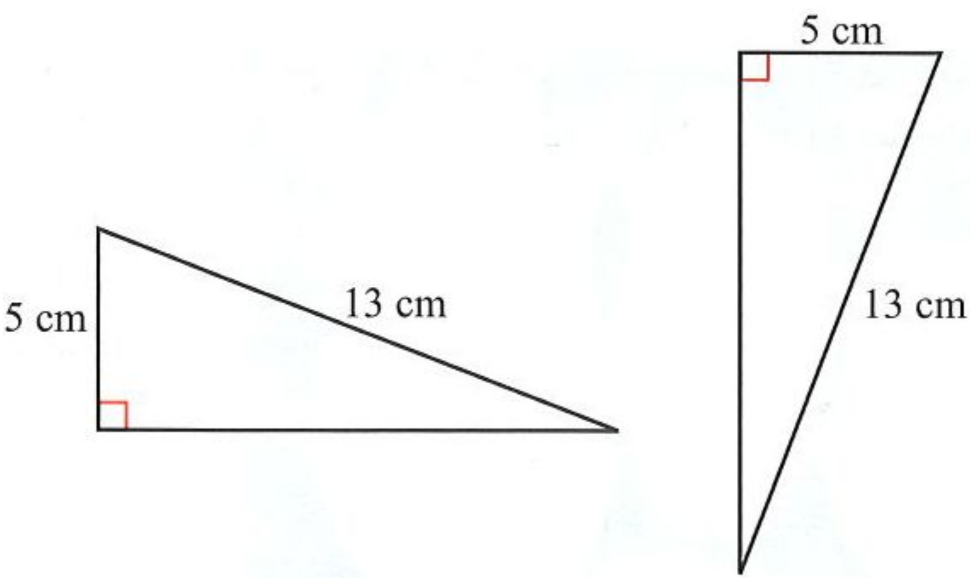
There are three pairs of equal sides.

Remember this as SSS – ***S***ide ***S***ide ***S***ide.



Two angles and the **included side** (the included side is the side that is placed between the two angles) are equal.

Remember this as ASA – ***A***ngle ***S***ide ***A***ngle.



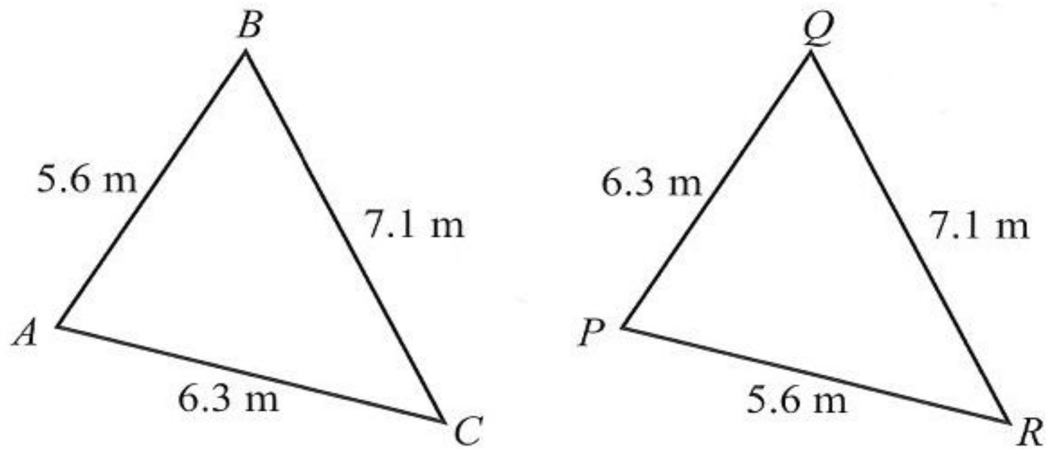
If you have right-angled triangles, the angle does not need to be included for the triangles to be congruent. The triangles must have the same length of hypotenuse and one other side equal.

Remember this as RHS – ***R***ight-angle ***S***ide ***H***ypotenuse.

Exercise 11.7

For each question show that the pair of shapes are congruent to one another. Explain each answer carefully and state clearly which of SAS, SSS, ASA or RHS you have used.

1



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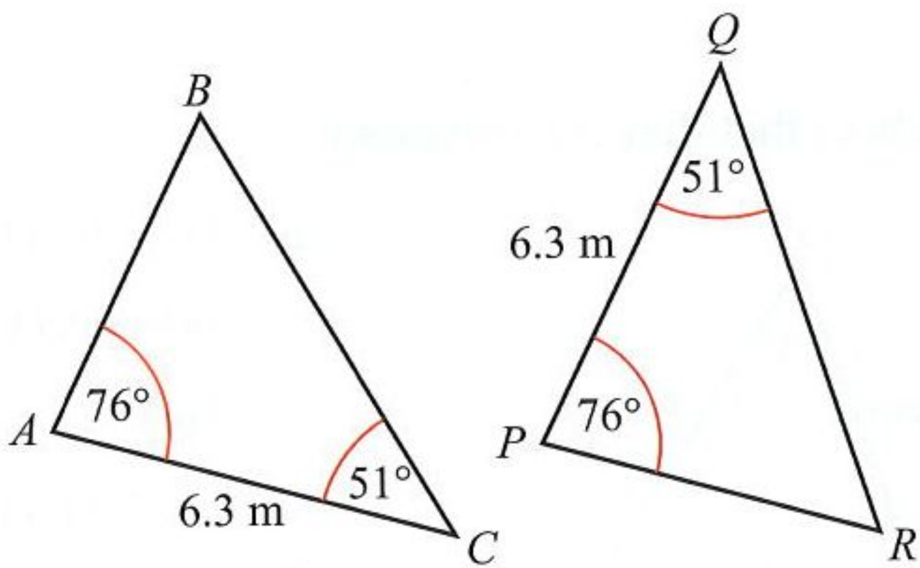

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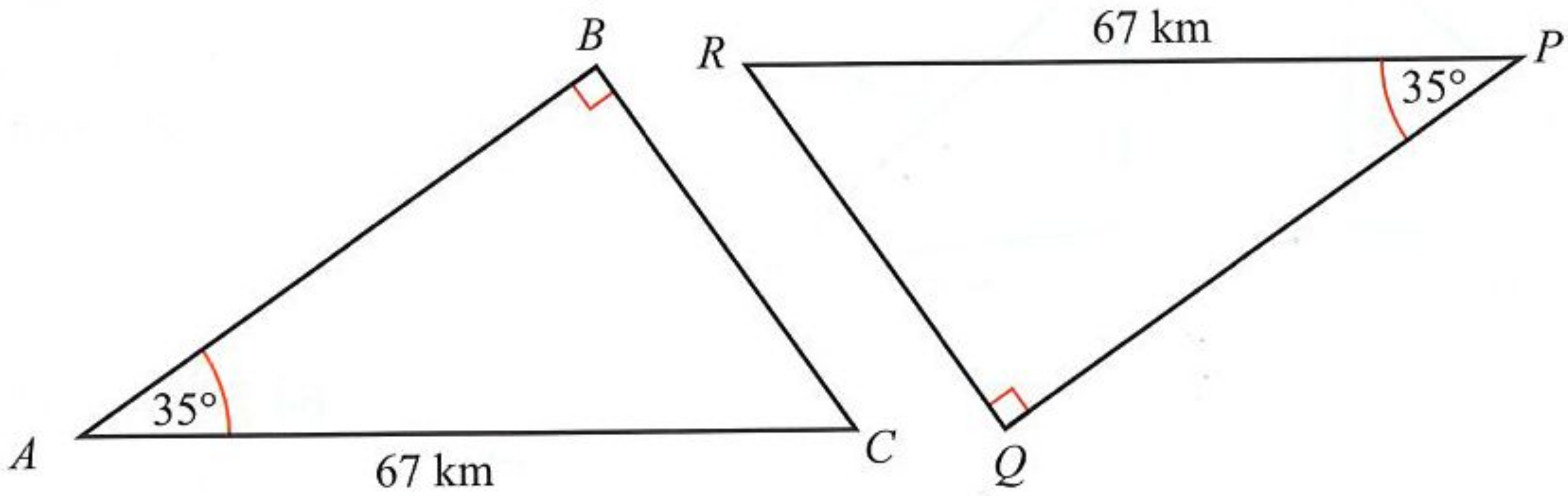

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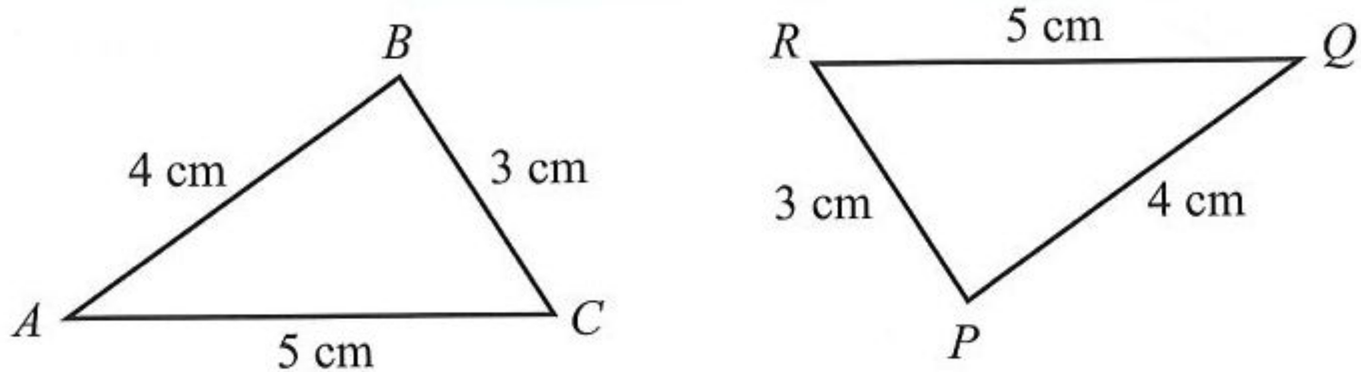

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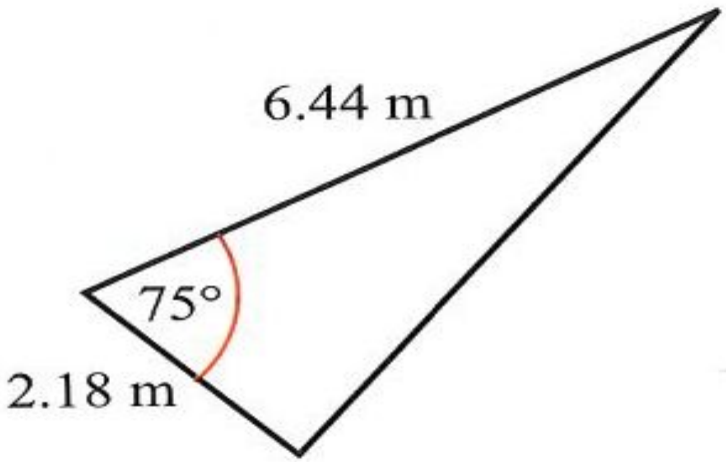

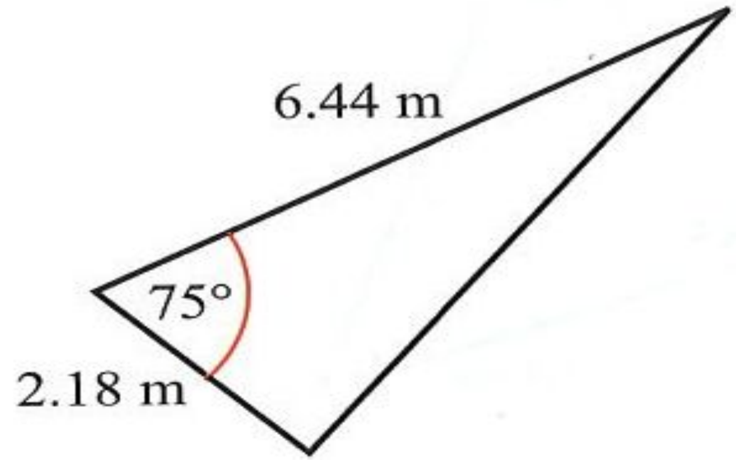

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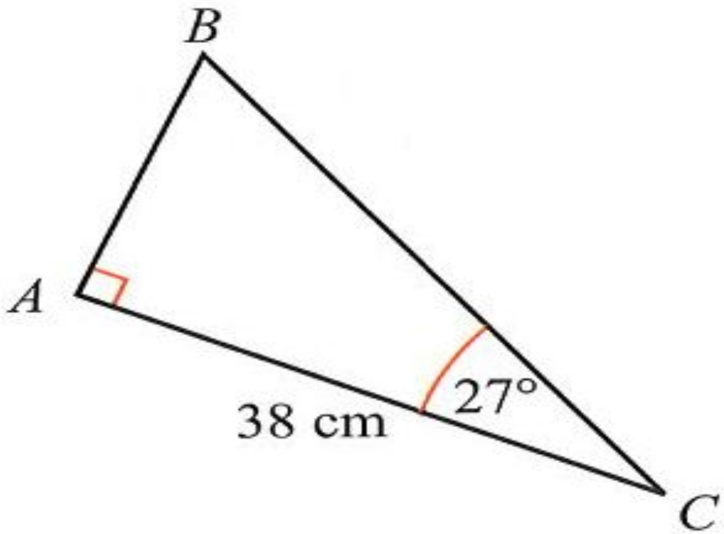
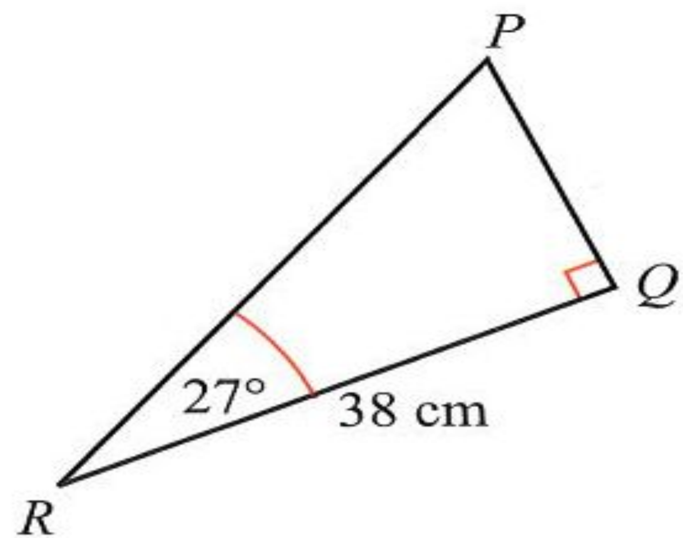




Summary

Do you know the following?

- The longest side of a right-angled triangle is called the hypotenuse.
- The square of the hypotenuse is equal to the sum of the squares of the two shorter sides of the triangle.
- Similar shapes have equal corresponding angles and the ratios of corresponding sides are equal.
- If shapes are similar and the lengths of one shape are multiplied by a scale factor of n:
  - then the areas are multiplied by a scale factor of n^2
  - and the volumes are multiplied by a scale factor of n^3.
- Congruent triangles are exactly equal to each other.

Are you able to ...?

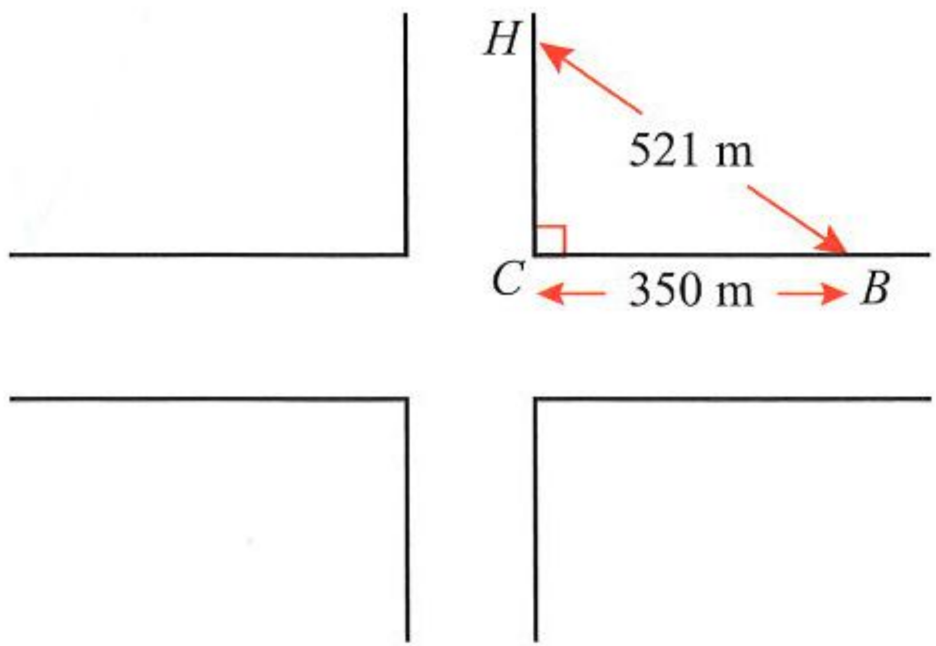
- use Pythagoras' theorem to find an unknown side of a right-angled triangle
- use Pythagoras' theorem to solve real-life problems
- decide whether or not two objects are mathematically similar
- use the fact that two objects are similar to calculate:
  - unknown lengths
  - areas or volumes
- decide whether or not two triangles are congruent.

Examination practice

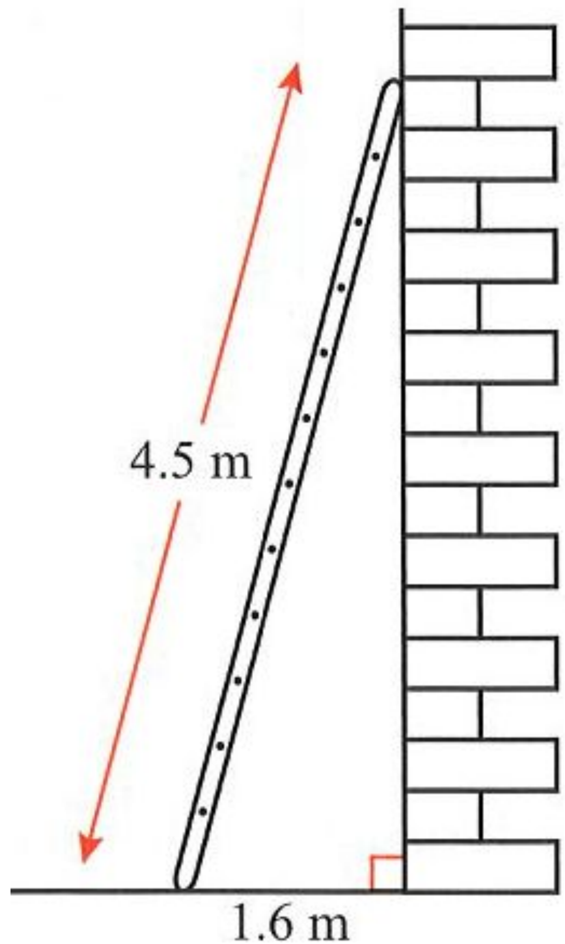
1 Mohamed takes a short cut from his home (H) to the bus stop (B) along a footpath HB.

How much further would it be for Mohamed to walk to the bus stop by going from H to the corner (C) and then from C to B?

Give your answer in metres.



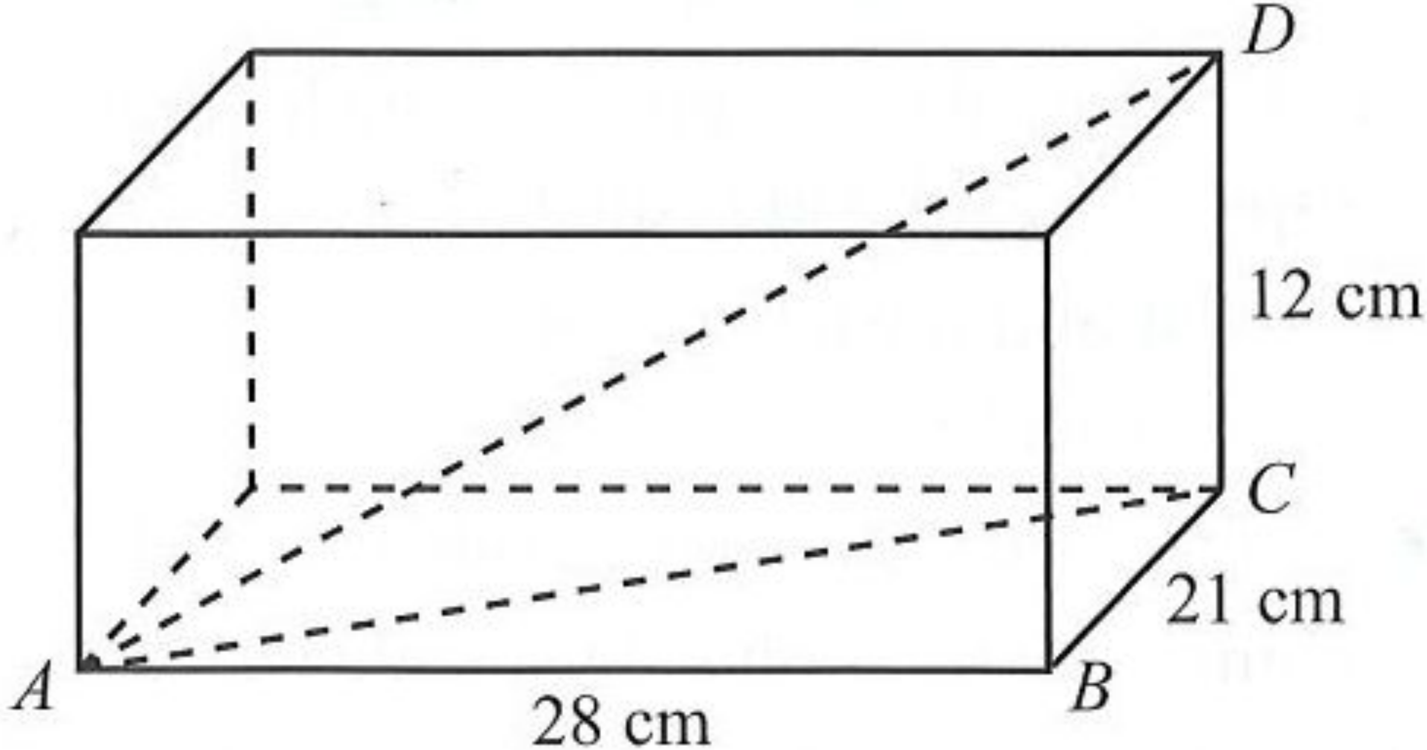
2 A ladder is standing on horizontal ground and rests against a vertical wall. The ladder is 4.5 m long and its foot is 1.6 m from the wall. Calculate how far up the wall the ladder will reach. Give your answer correct to 3 significant figures.



3 A rectangular box has a base with internal dimensions 21 cm by 28 cm, and an internal height of 12 cm. Calculate the length of the longest straight thin rod that will fit:

(a) on the base of the box

(b) in the box.



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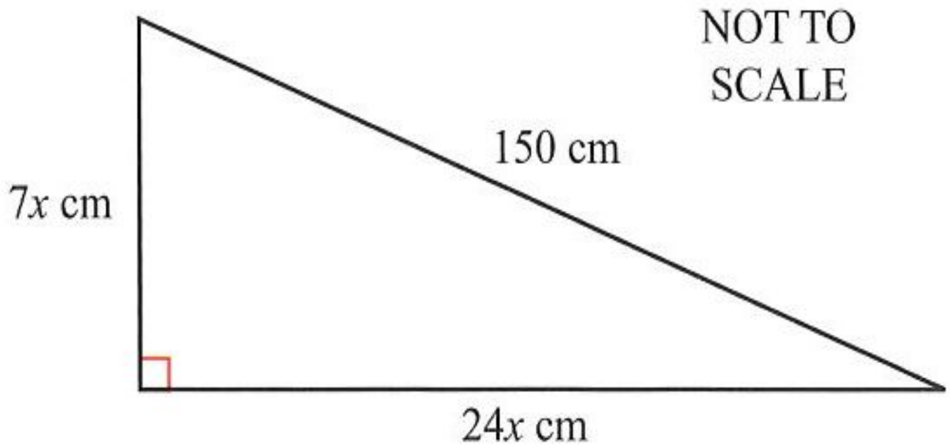

The right-angled triangle in the diagram has sides of length $7x$ cm , $24x$ cm and 150 cm.

(a) Show that $x^2 = 36$

(b) Calculate the perimeter of the triangle.