Error control.
Hamming code

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INFORMATION THEORY
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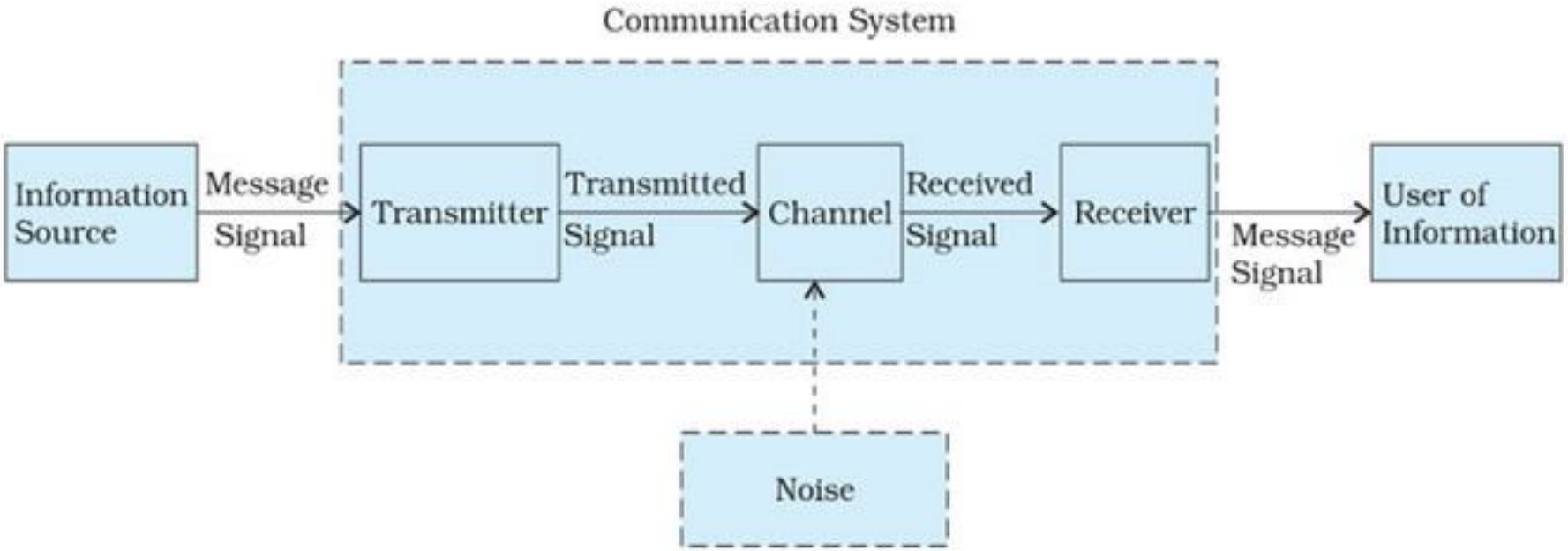
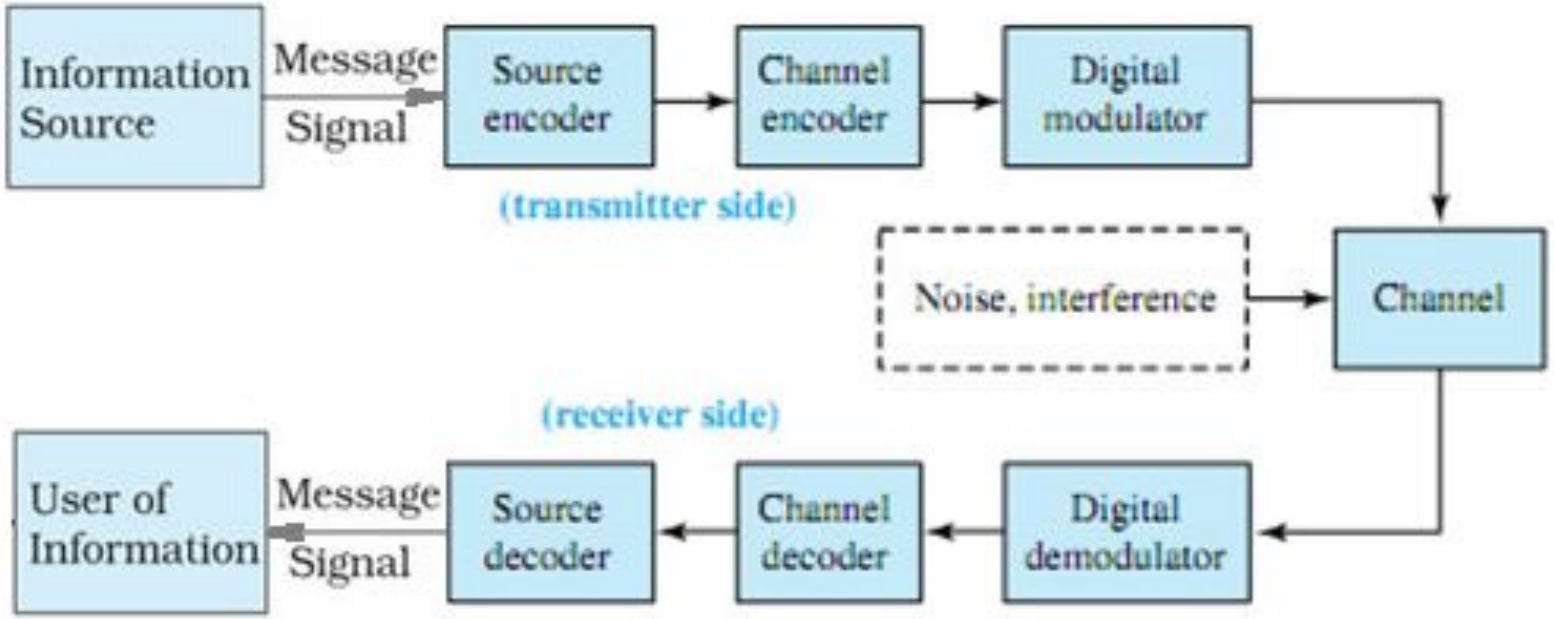
Communication system



Communication system



Error control

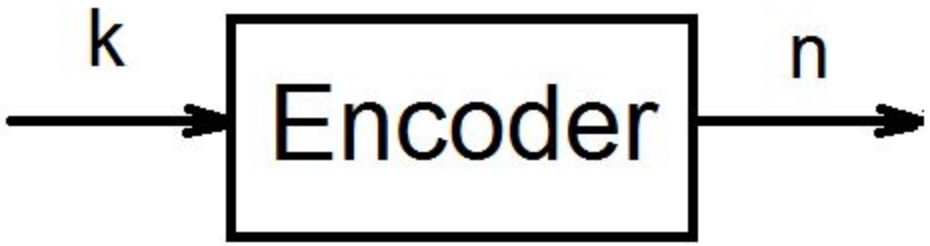
- In information theory and coding theory with applications in computer science and telecommunication **error control** are techniques that enable reliable delivery of digital data over unreliable communication channels.

Types of error control:

- **Error detection** is the detection of errors caused by noise or other impairments during transmission from the transmitter to the receiver.
- **Error correction** is the detection of errors and reconstruction of the original, error-free data.

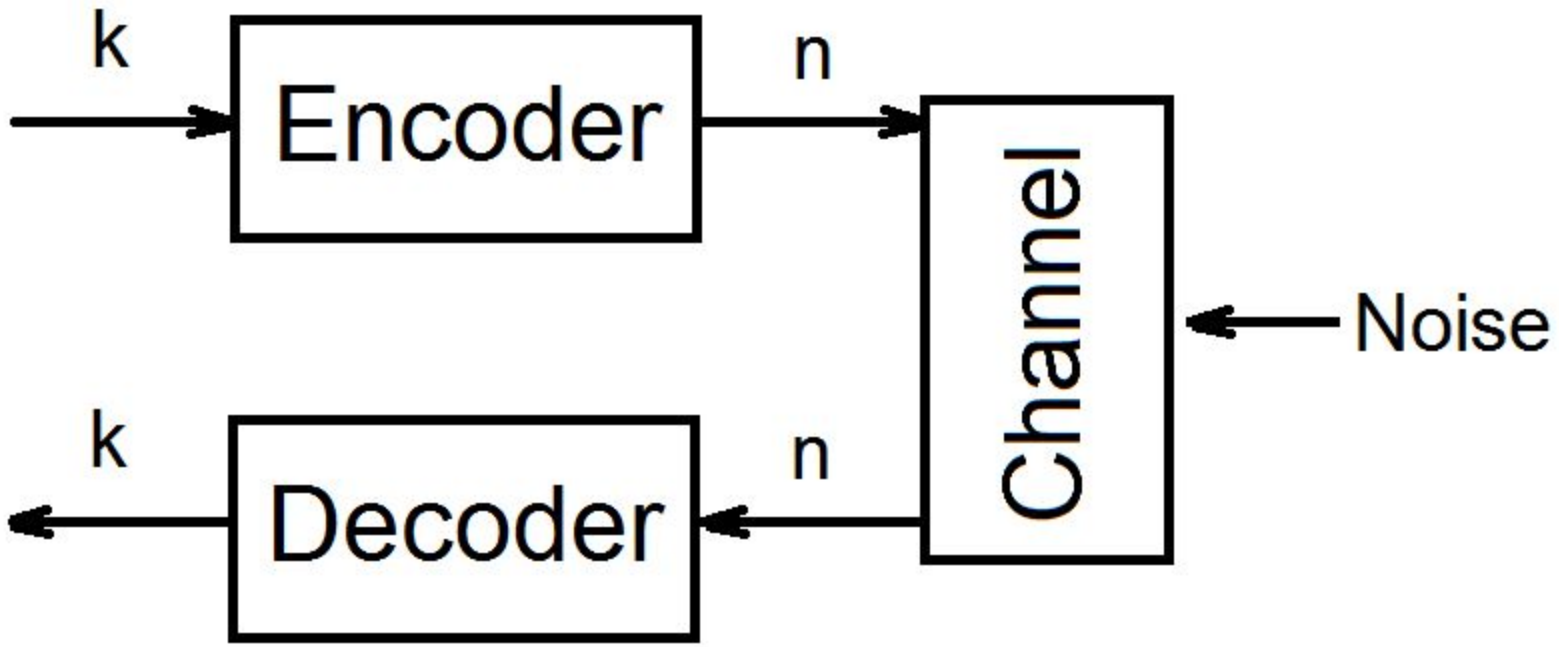
Main idea

- The general idea for achieving error detection and correction is to **add some redundancy** (i.e., some extra data) to a message, which receivers can use to check consistency of the delivered message, and to recover data determined to be corrupted.



$n > k$

Main idea



Error control

Different techniques of coding:

- Block code
- Convolutional code

Block codes

- k - length of each block before coding
- n - length of each block after coding
- such codes are denoted **(n, k)**
- as before **n > k**
- code rate:

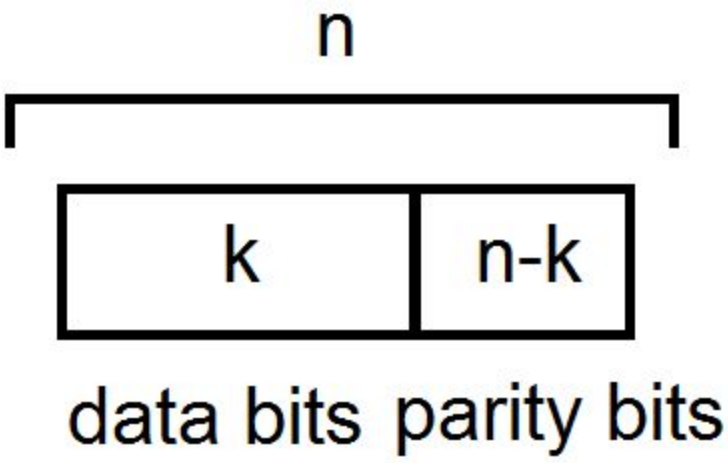
R=k/n

n > k

Block codes

Blocks:

- **Data bits** – information
- **Parity bits** – redundant



Example

Parity bit:

- **0** – if number of “1” in code combination is even
- **1** – if number of “1” in code combination is odd

Example

- 101 – code combination before coding
- 1010 - code combination after coding

Different combinations

Types of code combinations after a channel:

- **permissible (allowable) combinations** – code combination without error(s)
- **prohibited (not allowable) combinations** - code combination with error(s)

Detection or correction?

- **Hamming distance** between two strings of equal length is the number of positions at which the corresponding symbols are different.

- In another way, **Hamming distance** measures the minimum number of *substitutions* required to change one string into the other, or the minimum number of *errors* that could have transformed one string into the other.

Hamming distance: examples

- "karolin" and "kathrin" is 3.
- "karolin" and "kerstin" is 3.
- 1011101 and 1001001 is 2.
- 2173896 and 2233796 is 3.

- For binary strings a and b the Hamming distance is equal to the number of ones in **a XOR b**.

Example 1: Hamming distance=1

- When d=1 all code combinations are allowable and any mistake will cause the transition to another allowable code combination. Which means that no error can be detected. For example, when n=3, allowable combinations form the following set:

000	001	010	011	100	101	110	111

Example 2: Hamming distance=2

- With Hamming code distance d=2 there is no one from allowable code words which could transform to another. These are already allowable and not allowable combinations, so errors can be detected, but not corrected. For example, suppose n=3 as before.

allowable	000	011	101	110
not allowable	001	010	100	111

Example 3: Hamming distance=3

- In this example Hamming distance is enough for not only error detection, but also error correction. Every allowable combination has set of not allowable.

allowable	not allowable
000	001 010 100
111	011 101 110

Basic formulas

- Detect errors of multiplicity r:

  dmin >= r+1

- Correct errors of multiplicity s:

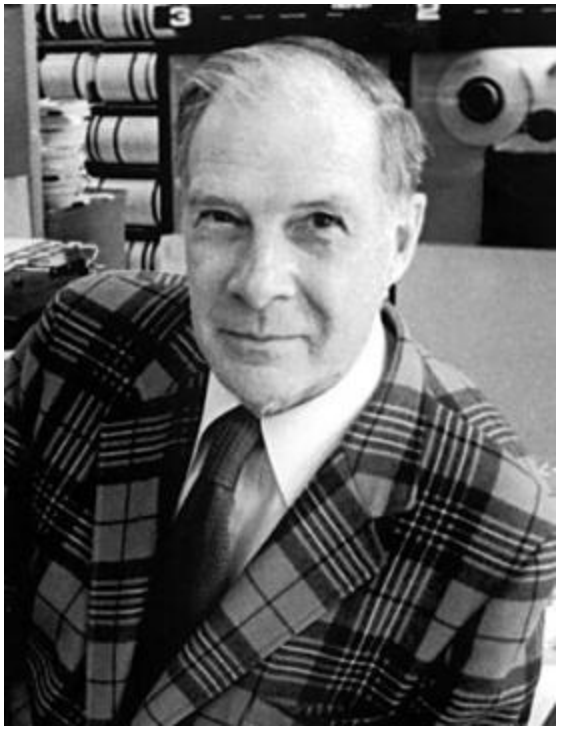
  dmin >= 2s+1

- To detect errors of multiplicity r and to correct errors of multiplicity s (general formula):

  dmin >= s+r+1

Hamming code

- **Hamming codes** are a family of linear error-correcting codes that generalize the Hamming (7,4) - code invented by Richard Hamming in 1950.



- Error detection &

  error correction

Main ideas

- Hamming was interested in two problems at once:
  - increasing the distance as much as possible
  - at the same time increasing the code rate as much as possible.

- The key to all of his systems was to have the parity bits overlap, such that they managed to check each other as well as the data.

Structure rule of Hamming codes

For each integer r ≥ 2 there is a code with block length **n = 2^r−1** and message length **k = 2^r−r−1**.

(n, k)=(2^r−1, 2^r−r−1)
For example, (7, 4), (15, 11), (31, 26), etc

Parity bits (r)	Total bits (n)	Data bits (k)	Name	Rate
3	7	4	Hamming(7,4)	4/7 ≈ 0.571
4	15	11	Hamming(15,11)	11/15 ≈ 0.733
5	31	26	Hamming(31,26)	26/31 ≈ 0.839

Hamming (7,4) code

- In 1950, Hamming introduced the (7,4) Hamming code. It encodes 4 data bits into 7 bits by adding three parity bits.

- (i1, i2, i3, i4) -> (i1, i2, i3, i4, r1, r2, r3),
  i – data bits and r – parity bits

- It can detect and correct single-bit errors – SEC (single-error correcting).

Encoding Hamming(7,4)

- The key thing about Hamming Codes is that any given bit is included in a unique set of parity bits.

- r1 = i1 XOR i2 XOR i3
- r2 = i2 XOR i3 XOR i4
- r3 = i1 XOR i2 XOR i4

Decoding (7,4)

- Decoder gets a codeword (i1, i2, i3, i4, r1, r2, r3), where every bit can be an error bit (including data and parity bits).
- The pattern of errors, called the **error syndrome**, identifies the bit in error.

- S1 = r1 XOR i1 XOR i2 XOR i3
- S2 = r2 XOR i2 XOR i3 XOR i4
- S3 = r3 XOR i1 XOR i2 XOR i4

- **S = (s1, s2, s3)** - error syndrome

Decoding (7,4)

Error syndrome	Error bit
000	No error
001	r3
010	r2
011	i4
100	r1
101	i1
110	i3
111	i2

Example 1: an error in data bit

- Combination before encoding: 1001
- Combination after encoding: 1001**110**
- Single random error: i4
- Combination with error: 100**0**110
- Decoding syndrome: 011 -> i4
- Decoding (error correction): 100**1**110
- After decoding 1001

Example 2: an error in parity bit

- Combination before encoding: 1001
- Combination after encoding: 1001**<u>110</u>**
- Single random error: r2
- Combination with error: 10011**<u>0</u>**0
- Decoding syndrome: 010 -> r2
- Decoding (error correction): 10011**<u>1</u>**0
- After decoding 1001

General algorithm

- To compare different approaches consider Hamming(7,4) as example.
- However this general algorithm can be used for any length codewords.

- In the example:

  Input: 4-bit code word x1…x4

Input codeword

- Row 1 – number of position in the codeword
- Row 2 – bit notation
- Row 3 – bit value

  (so example codeword is “1001”)

1	2	3	4
x_1	x_2	x_3	x_4
1	0	0	1

Number of parity bits

- In general, the number of parity bits in the codeword is equal to the binary logarithm of the number of bits of the codeword (including parity bits) in rounded up to the nearest integer:

$$\mathbf{r \approx \log_2(n)}$$

- In the example, $r = \log_2(7) \approx 3$

Addition of parity bits

- Add parity bits r0,r1,r2
- Number of positions are integer powers of 2: 0, 1, 2, etc: 2^0, 2^1, 2^2, etc.
- Now we have 7-bit word with 4 data bits and 3 parity bits.
- Initially parity bits are set to zero.

1	2	3	4	5	6	7
r_0	r_1	x_1	r_2	x_2	x_3	x_4
0	0	1	0	0	0	1

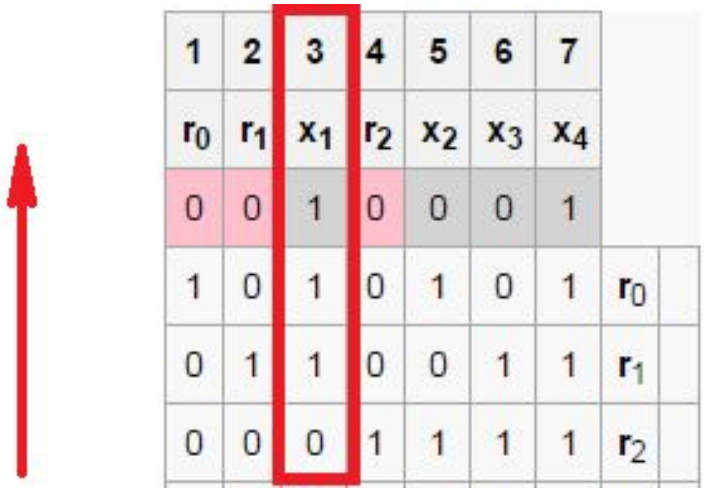
Transformation matrix

- Add to table 3 rows (number of parity bits) with a **transformation matrix**.
- Each row is one parity bit (top down), each column is one bit of codeword.

1	2	3	4	5	6	7	
r_0	r_1	x_1	r_2	x_2	x_3	x_4	
0	0	1	0	0	0	1	
1	0	1	0	1	0	1	r_0
0	1	1	0	0	1	1	r_1
0	0	0	1	1	1	1	r_2

Transformation matrix

- Each column of a transformation matrix is a binary number of this column, but bit order is reverse: the least significant bit is on the top line, a senior - at the bottom.
- For example, 3rd column is “110” corresponding to the binary representation of the number “3”: 011.



1	2	3	4	5	6	7	
r_0	r_1	x_1	r_2	x_2	x_3	x_4	
0	0	1	0	0	0	1	
1	0	1	0	1	0	1	r_0
0	1	1	0	0	1	1	r_1
0	0	0	1	1	1	1	r_2

Calculation of parity bits

r0 = (1·0+0·0+1·1+0·0+1·0+0·0+1·1) mod 2 = 2 mod 2 = 0
r1 = (0·0+1·0+1·1+0·0+0·0+1·0+1·1) mod 2 = 2 mod 2 = 0
r2 = 0·0+0·0+0·1+1·0+1·0+1·0+1·1 =1

- The resulting control bits inserted into codeword instead of standing there before zeros.
- Hamming coding is completed. The resulting code word - 0011001

1	2	3	4	5	6	7	
r_0	r_1	x_1	r_2	x_2	x_3	x_4	
0	0	1	0	0	0	1	
1	0	1	0	1	0	1	r_0
0	1	1	0	0	1	1	r_1
0	0	0	1	1	1	1	r_2

1	2	3	4	5	6	7		
r_0	r_1	x_1	r_2	x_2	x_3	x_4		
0	0	1	1	0	0	1		
1	0	1	0	1	0	1	r_0	0
0	1	1	0	0	1	1	r_1	0
0	0	0	1	1	1	1	r_2	1

Decoding

- Algorithm of decoding is absolutely identical to encoding algorithm.
- Goal of decoding is get a **syndrome matrix**.
- As before the syndrome matrix (0,0,0) indicates a codeword without error, any other – with error.
- For example, change one of the bits (6-th bit) to show an error and its correction.

Decoding

s0 = (1*0+0*0+1*1+0*1+1*0+0*1+1*1) mod 2 = 2 mod 2 = 0
s1 = (0*0+1*0+1*1+0*1+0*0+1*1+1*1) mod 2 = 3 mod 2 = 1
s2 = (0*0+0*0+0*1+1*1+1*0+1*1+1*1) mod 2 = 3 mod 2 = 1

Syndrome matrix is (0,1,1)

1	2	3	4	5	6	7		
r_0	r_1	x_1	r_2	x_2	x_3	x_4		
0	0	1	1	0	1	1		
1	0	1	0	1	0	1	s_0	0
0	1	1	0	0	1	1	s_1	1
0	0	0	1	1	1	1	s_2	1

Decoding

- Syndrome matrix is a binary number of error position.
- In example s = 011 and decimal representation of “110” is “6”, so error position = 6.

1	2	3	4	5	6	7		
r_0	r_1	x_1	r_2	x_2	x_3	x_4		
0	0	1	1	0	1	1		
1	0	1	0	1	0	1	s_0	0
0	1	1	0	0	1	1	s_1	1
0	0	0	1	1	1	1	s_2	1

Example of general algorithm for Hamming (15,11)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
r_0	r_1	x_1	r_2	x_2	x_3	x_4	r_3	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
0	0	1	0	0	0	1	0	0	0	1	0	1	1	1	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	r_0
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	r_1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	r_2
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	r_3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	r_4

References

- **Arndt C.**
  Information Measures: Information and its Description in Science and Engineering.
- **Thomas Cover.** Elements Of Information Theory.