Production Management

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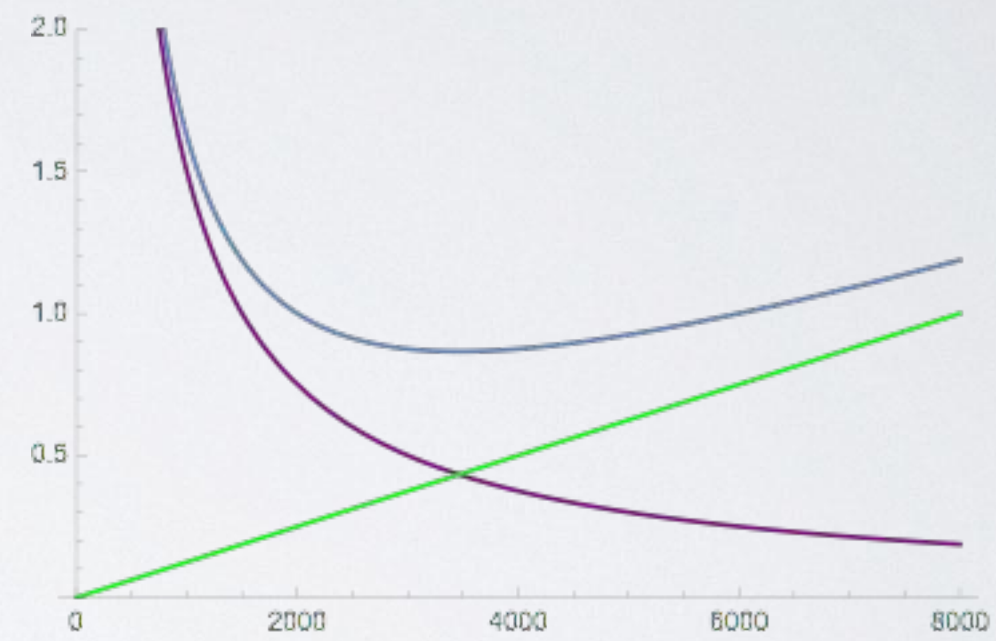
EOQ Model

Economic Order Quantity is the order quantity that minimizes total holding and ordering costs for the year.



Overview

Economic order quantity is the order quantity that minimizes total inventory holding costs and ordering costs. It is one of the oldest classical production scheduling models.

EOQ applies only when demand for a product is constant over the year and each new order is delivered in full when inventory reaches zero. There is a fixed cost for each order placed, regardless of the number of units ordered. There is also a cost for each unit held in storage, commonly known as holding cost, sometimes expressed as a percentage of the purchase cost of the item.

EOQ search to determine the optimal number of units to order to minimize the total cost associated with the purchase, delivery and storage of the product.

The required parameters to the solution are the total demand for the year, the purchase cost for each item, the fixed cost to place the order and the storage cost for each item per year. Note that the number of times an order is placed will also affect the total cost, though this number can be determined from the other parameters.

Underlying assumptions

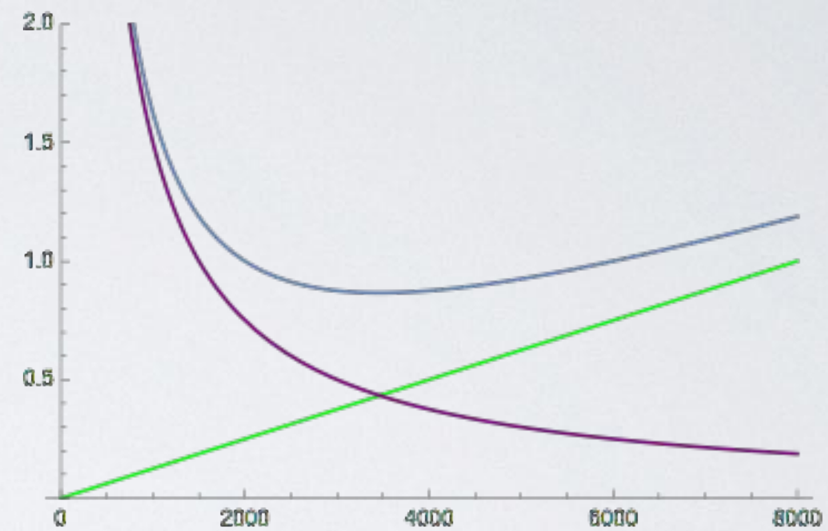
1. The ordering cost is constant.
2. The rate of demand is known, and spread evenly throughout the year.
3. The lead time is fixed.
4. The purchase price of the item is constant i.e. no discount is available
5. The replenishment is made instantaneously, the whole batch is delivered at once.
6. Only one product is involved.

Variables

- *c* : purchase price, unit production cost
- *Q* : order quantity
- *Q** : optimal order quantity
- *D* : annual demand quantity
- *K* : fixed cost per order, setup cost (not per unit, typically cost of ordering and shipping and handling. This is not the cost of goods)
- *H* : annual holding cost per unit, also known as carrying cost or storage cost (capital cost, warehouse space, refrigeration, insurance, etc. usually not related to the unit production cost)

The total cost function

$$TC(Q) = cD + \frac{DK}{Q} + \frac{HQ}{2}$$

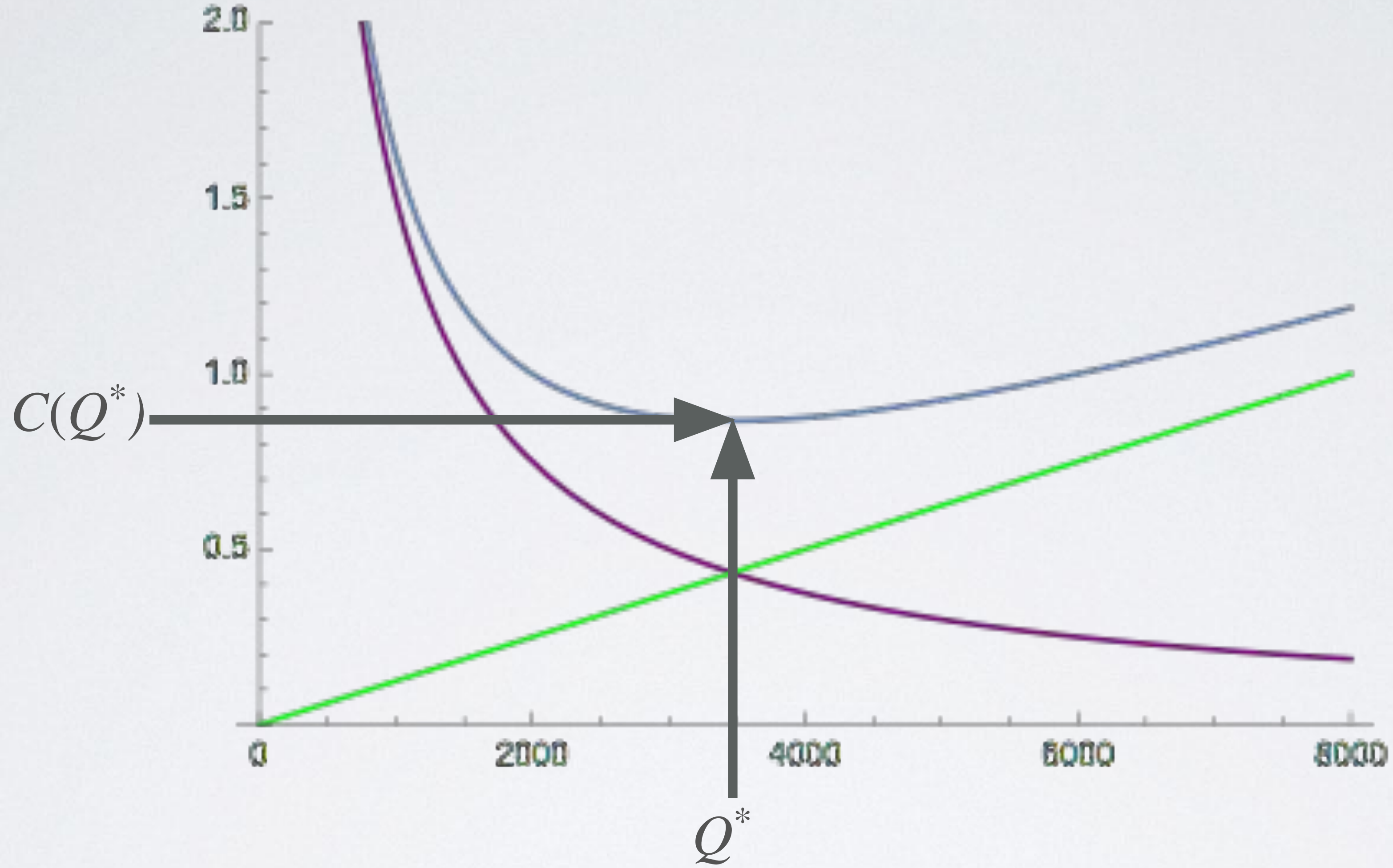


To obtain the optimal value of Q^*, we compute…

$$\frac{\partial TC(Q)}{\partial Q} = 0$$

$$Q^* = \sqrt{\frac{2DK}{H}}$$

Optimal value of Q



At optimal value of Q^*, we have the same holding cost and order cost…

Price of the item based on the quantity ordered

It is often possible to obtain price discounts when quantities ordered are below certain thresholds

- 10 € per item when the order quantity is in interval]0 ; 400[
- 9,5 € per item when the order quantity is in interval [400 ; 600[
- 9 € per item when the order quantity is in interval [600 ; inf[.

Formally, order are partitioned to intervals [a_i, b_i] based on the price of the item

Interval	Q^*	Q_{opt}	c_i	$R(Q_{opt})$
] 0 ; 400 [	?	?	10,00 €	?
[400 ; 600 [		?	9,50 €	?
[600 ; oo [		?	9,00 €	?

Algorithm

- Compute Q^*
- For each interval :
  - Determine Q_{opt}
  - Compute R(Q_{opt})
- Keep Q_{opt} giving min($R(Q_{opt})$)

Order cost dependent of quantity

It is often possible to obtain price discounts when quantities ordered are below certain thresholds

- K = 1000 €, if $Q \leq 400$ (road transport)
- K = 1500 €, if $Q > 400$ (rail transport)

D =1000 item / day

H = 1 € / item / day

For each intervals [a_i, b_i], order cost is different…

Interval i	K	Q^*	Q_{opt}	$C(Q_{opt})$
] 0 ; 400]	1000 €	?	?	?
] 400 ; 600 [	1500 €	?	?	?

Algorithm

- For each interval :
  - Compute Q^* based on K
  - Determine Q_{opt}
  - Compute $C(Q_{opt})$
- Keep Q_{opt} giving min($C(Q_{opt})$)

Degressive prices based on quantity

It is often possible to obtain price discounts when quantities ordered are below certain thresholds

- 10 € per item in interval]0 ; 400[
- 9,5 € per item in interval [400 ; 600[
- 9 € per item in interval [600 ; inf[.

Formally, order are partitioned to intervals [a_i, b_i] based on the price of the item

Interval	K_i'	Q_i*	Q_{opt}	$R(Q_{opt})$
] 0 ; 400 [	?	?	?	?
[400 ; 600 [	?	?	?	?
[600 ; oo [	?	?	?	?

Algorithm

- Compute $K'_i = K$
- For each interval :
  - Determine Q_i*
  - Compute Q_{opt} & $R(Q_{opt})$
  - $K'_i = K'_i + b_i(c_{i+1} - c_i)$
- Keep Q_{opt} giving $\min(R(Q_{opt}))$

Periodic replenishment

The idea is to replenish the stock level up to the maximum quantity at fixed period

Model based on the service rate

The criterion is to supply the demand. There are two indicators:

- α: Probability to meet demand without breaking
- β: Ratio between the number of items supplied and demand

Exemple

A pharmacist sells every day 6-9 boxes of some infant vaccine. The vaccine should be kept refrigerated. Space is limited and replenishment is daily. The pharmacist decides to fix the reorder point 7 boxes.

x	px	F(x)
6	35%	35%
7	50%	85%
8	10%	95%
9	5%	100%

$$\frac{\partial TC(Q)}{\partial Q} = 0$$

$$\alpha = F(S)$$

$$P(S) = \sum_{x=S}^{\infty} (x-S)\,px = G(S) - S\left(1 - F(S)\right)$$

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Stock with zero rotation: the newsagent

We consider that the cost of holding C_P is equal to the cost of unsold for an article.
The cost of shortage C_R is proportional to the number of missing items.

Example 1

A newsagent buys every day a fixed amount of a magazine. It pays each copy 2 € and sold 3 €, representing a margin of 1 €.
If demand exceeds its stock, it finds a shortfall of 1 €, so C_R = 1. Any unsold magazine was bought 1.5 € per press group which causes a loss of 0.5 €, so C_P = 0,5.

Stock with zero rotation: the newsagent

Model

Let px the probability that the demand *X* is equal to *x* items on period *T* and *S* is the desired level of stock.

if x ≤ S there is (S-x) unsold, representing a cost of CP(S-x)

Si x > S, (x-S) items are missing, representing a cost of CR(x-S)

A request of x articles having a probability px of occurring, the average total cost is given by:

$$C(S)=C_P\cdot T\cdot\sum_{x=0}^{S}\left(S-\frac{x}{2}\right)px+C_P\cdot T\cdot\sum_{x=S+1}^{n}\frac{S^2}{2x}px+C_R\cdot\sum_{x=S+1}^{n}(x-S)px \qquad \text{Eq. 1}$$

Warning : it is for x ≤ S and x > S. If we have x < S et x ≥ S, we obtain:

$$C(S)=C_P\cdot\sum_{x=0}^{S}(S-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S)px \qquad \text{Eq. 2}$$

$$C(S+1)-C(S)=\left(C_P\cdot\sum_{x=0}^{S}(S+1-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S-1)px\right)-(\ )$$

$$C(S+1)-C(S)=\left(C_P\cdot\sum_{x=0}^{S}(S+1-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S-1)px\right)-\left(C_P\cdot\sum_{x=0}^{S}(S-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S)px\right)$$

Stock with zero rotation: the newsagent

With the cumulative density function F(S), we obtain

$$C(S+1)-C(S)=\left(C_P\cdot\sum_{x=0}^{S}(S+1-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S-1)px\right)-\left(C_P\cdot\sum_{x=0}^{S}(S-x)px+C_R\cdot\sum_{x=S+1}^{n}(x-S)px\right)$$

$$C(S+1)-C(S)=C_P\cdot\sum_{x=0}^{S}px-C_R\cdot\sum_{x=S+1}^{n}px$$

$$C(S+1)-C(S)=C_P\cdot F(S)-C_R\cdot(1-F(S))$$

Let $G(S)=\sum_{x=S+1}^{n}\frac{px}{x}$

$$C(S+1)-C(S)=(C_p\cdot T+CR)\cdot(1-F(S))+C_p\cdot T\left[1+\frac{2S+1}{2}G(S)\right] \longrightarrow L(S)<\rho$$

$$C(S+1)<C(S) \longrightarrow L(S)=\rho$$

$$C(S+1)=C(S) \longrightarrow L(S)>\rho$$

If $L(S^*-1)<\rho<L(S^*)$ — S^* is the only optimal solution

If $L(S^*-1)=\rho<L(S^*)$ — (S^*-1) and S^* are the two optimal solutions

Stock with zero rotation: the newsagent

Exercise

The newsagent observed over 10 weeks the number of newspapers sold.
Every day, he sold between 20 and 60 magazines

Number of magazines	n
] 20 , 25]	2
] 25 , 30]	4
] 30 , 35]	8
] 35 , 40]	10
] 40 , 45]	8
] 45 , 50]	8
] 50 , 55]	7
] 55 , 60]	3

$C_P = 3$ $\quad C_R = 2$

What is the optimal level of inventory?

Replenishment with C_P, C_R based on time and number of articles

Let C_P and C_R holding cost and shortage cost for one item for a period.
Let x the observed demand during the period T.

Model

1. If $x \leq S$, there is no out of stock

   By approximating the demand with a line, we obtain:

$$C_R(x-S)$$

1. if $x \geq S$, after a time T_1, there is out of short

   Holding costs amounted to

$$C_P \cdot \frac{S}{2} \cdot T1$$

   Shortage costs amounted to

$$C_P \cdot \frac{S}{2} \cdot T_1$$

   Thanks to Chasles relation a :

$$C_R \cdot \frac{x-S}{2} \cdot T_2$$

   Given a total cost

$$C_P \cdot \frac{S}{2} \cdot T_1$$

Replenishment with C_P, C_R based on time and number of articles

Since T is a given, the overall cost will be used per unit of time:

$$C_R \cdot \frac{x-S}{2} \cdot T_2$$

With probability….

$$\gamma(S) = \frac{C(S)}{T}$$

And, if we consider instead x < S et x ≥S, we obtain:

$$\gamma(S) = C_P \cdot \sum_{x=0}^{S}\left(S - \frac{x}{2}\right)px + C_P \cdot \sum_{x=S+1}^{n} \frac{S^2}{2x} px + C_R \cdot \sum_{x=S+1}^{n} \frac{(x-S)^2}{2x} px$$

$$\gamma(S+1) = C_P \cdot \sum_{x=0}^{S}\left(S + 1 - \frac{x}{2}\right)px + C_P \cdot \sum_{x=S+1}^{n} \frac{(S+1)^2}{2x} px + C_R \cdot \sum_{x=S+1}^{n} \frac{(x-S-1)^2}{2x} px$$

Replenishment with C_P, C_R based on time and number of articles

$$\gamma(S+1) = C_P \cdot \sum_{x=0}^{S} \left(S + 1 - \frac{x}{2} \right) px + C_P \cdot \sum_{x=S+1}^{n} \frac{(S+1)^2}{2x} px + C_R \cdot \sum_{x=S+1}^{n} \frac{(x-S-1)^2}{2x} px$$

With the cumulative probability function F(S), the relation can be expressed by:

$$\gamma(S+1) - \gamma(S) = C_P \cdot \sum_{x=0}^{S} px + C_P \cdot \sum_{x=S+1}^{n} \frac{2S+1}{2x} px - C_R \cdot \sum_{x=S+1}^{n} \frac{2x-2S-1}{2x} px$$

$$\gamma(S+1) - \gamma(S) = C_P \cdot \left[F(S) + \frac{2S+1}{2} \sum_{x=S+1}^{n} \frac{px}{x} \right] - C_R \cdot \left[1 - F(S) - \frac{2S+1}{2} \sum_{x=S+1}^{n} \frac{px}{x} \right]$$

Let $\gamma(S+1) - \gamma(S) = C_P \cdot \sum_{x=0}^{S} px + C_P \cdot \sum_{x=S+1}^{n} \frac{2S+1}{2x} px - C_R \cdot \sum_{x=S+1}^{n} \frac{2x-2S-1}{2x} px$

$$G(S) = \sum_{x=S+1}^{n} \frac{px}{x}$$

$$\gamma(S+1) - \gamma(S) = (C_P + C_R) \cdot \left[F(S) + \frac{2S+1}{2} \sum_{x=S+1}^{n} \frac{px}{x} \right] - C_R$$

Replenishment with C_P, C_R based on time and number of articles

$L(S)$ is a positive function, So, we obtain a similar result to the previous problem...

$$C(S+1)-C(S)=(C_p \cdot T+CR)\cdot(1-F(S))+C_p \cdot T\left[1+\frac{2S+1}{2}G(S)\right] \longrightarrow H(S)>\theta$$

$$C(S+1)<C(S) \longrightarrow H(S)=\theta$$

$$C(S+1)=C(S) \longrightarrow H(S)<\theta$$

Moreover, $L(S)$ is strictly increasing, so:

If $H(S^*-1)>\theta>H(S^*)$ — S^* is the only optimal solution

If $H(S^*-1)=\theta>H(S^*)$ — (S^*-1) and S^* are the two optimal solutions

Replenishment with C_P, C_R based on time and number of articles

Exercise

A workpiece is supplied every 15 days. It is estimated that the shortage cost for a period is equivalent to 20 holding cost (over the same period).
Over 15 days, the demand varie from 20 to 27.

x , S	px
20	4%
21	8%
22	16%
23	20%
24	16%
25	16%
26	14%
27	6%

Replenishment with C_P, C_R based on time and number of articles

Exercise

A workpiece is supplied every 15 days. It is estimated that the shortage cost for a period is equivalent to 20 holding cost (over the same period).
Over 15 days, the demand varie from 20 to 27.

x , S	px	F(x)	px/x	G(S)	L(S)
20	4%	4%	2,00E-03	4,05E-02	0,869
21	8%	12%	3,81E-03	3,66E-02	0,908
22	16%	28%	7,27E-03	2,94E-02	0,941
23	20%	48%	8,70E-03	2,07E-02	0,966
24	16%	64%	6,67E-03	1,40E-02	0,983
25	16%	80%	6,40E-03	7,61E-03	0,994
26	14%	94%	5,38E-03	2,22E-03	0,999
27	6%	100%	2,22E-03	0,00E+00	1,000

$$\rho = \frac{20}{20+1} \approx 0{,}9523$$

$$\rho = \frac{20}{1+20} \approx 0{,}9523$$

$S^* = 23$ unités

Replenishment with C_R based on number of articles

Model

This model differs from the previous by the computation of the shortage cost

When $x \geq S$, after a period T_I, ther is out of stock

Shortage cost: $C_R(x-S)$

In terms of probability, total cost will be:

$$C_R(x-S)$$

$$C(S+1)-C(S)=(C_P \cdot T + CR)\cdot(1-F(S))+C_P \cdot T \cdot \sum_{x=S+1}^{n} \frac{S^2}{2x} px + C_R \cdot \sum_{x=S+1}^{n} (x-S) px$$

Let $\left[1+\frac{2S+1}{2}G(S)\right]$

$$H(S)=\frac{1-F(S)}{1+\left(S+\frac{1}{2}G(S)\right)}$$

Replenishment with C_R based on number of articles

$H(S)$ is a positive function

$$H(S) = \frac{1 - F(S)}{1 + \left(S + \frac{1}{2} G(S)\right)}$$

$$C(S+1) - C(S) = (C_p \cdot T + CR) \cdot (1 - F(S)) + C_p \cdot T \left[1 + \frac{2S+1}{2} G(S)\right] \longrightarrow C(S+1) < C(S)$$

$$C(S+1) < C(S) \longrightarrow H(S) > \theta$$

$$C(S+1) = C(S) \longrightarrow H(S) = \theta$$

If $H(S^* - 1) \geq \theta > H(S^*)$ S^* is the only optimal solution

If $H(S^* - 1) > \theta > H(S^*)$ (S^*-1) and S^* are the two optimal solutions

Replenishment with C_R based on number of articles

Exercise

A shop sells every week between 20 and 25 computers. Replenishment is done every Thursday night. With competition strong, anyone who can not take his product will buy directly from a competitor. In this case, the distributor estimates its loss to 200 €. On the other hand, the shop buys a computer 2000 € and estimates the annual holding cost to 25% of the purchase price.

x , S	px
20	6%
21	15%
22	40%
23	30%
24	5%
25	4%

Replenishment with C_R based on number of articles

Exercise

A shop sells every week between 20 and 25 computers. Replenishment is done every Thursday night. With competition strong, anyone who can not take his product will buy directly from a competitor. In this case, the distributor estimates its loss to 200 €. On the other hand, the shop buys a computer 2000 € and estimates the annual holding cost to 25% of the purchase price.

x , S	px	F(X)	px/x	G(S)	
20	6%	6%	0,0030	0,0421	0,5048
21	15%	21%	0,0071	0,0349	0,4513
22	40%	61%	0,0182	0,0167	0,2834
23	30%	91%	0,0130	0,0037	0,0828
24	5%	96%	0,0021	0,0016	0,0385
25	4%	100%	0,0016	0,0000	0,0000

$C_p = 500 € = 9{,}62 €/\text{semaine}$

$$C_p = 500 €$$

$C_p = 500 €$

$C_R = 200 €$